# 8

# BINOMIAL THEOREM

Suppose you need to calculate the amount of interest you will get after 5 years on a sum of money that you have invested at the rate of 15% compound interest per year. Or suppose we need to find the size of the population of a country after 10 years if we know the annual growth rate. A result that will help in finding these quantities is the **binomial theorem.** This theorem, as you will see, helps us to calculate the rational powers of any real binomial expression, that is, any expression involving two terms.

The binomial theorem, was known to Indian and Greek mathematicians in the 3rd century B.C. for some cases. The credit for the result for natural exponents goes to the Arab poet and mathematician Omar Khayyam (A.D. 1048-1122). Further generalisation to rational exponents was done by the British mathematician Newton (A.D. 1642-1727).

There was a reason for looking for further generalisation, apart from mathematical interest. The reason was its many applications. Apart from the ones we mentioned at the beginning, the binomial theorem has several applications in probability theory, calculus, and in approximating numbers like $(1.02)^7$, $3^{1/5}$, etc. We shall discuss a few of them in this lesson. Before discussing Binomial Theorem, we shall introduce the concept of Principle of Mathematical Induction, which we shall be using in proving the Binomial Theorem for a positive integral index. This principle is also useful in making generalisations from particular statements/results.

## OBJECTIVES

**After studying this lesson, you will be able to:**

- state the Principle of (finite) Mathematical Induction;
- verify the truth or otherwise of the statement P(n) for n = 1;
- verify if P(k+1) is true, assuming that P(k) is true;
- use principle of mathematical induction to establish the truth or otherwise of mathematical statements;

- state the binomial theorem for a positive integral index and prove it using the principle of mathematical induction;
- write the binomial expansion for expressions like $(x+y)^n$ for different values of $x$ and $y$ using binomial theorem;
- write the general term and middle term (s) of a binomial expansion;
- write the binomial expansion for negative as well as for rational indices;
- apply the binomial expansion for finding approximate values of numbers like $\sqrt[3]{9}, \sqrt{2}, \sqrt[3]{3}$ etc; and
- apply the binomial expansion to evaluate algebraic expressions like $\left(3-\frac{5}{x}\right)^7$, where x is so small that $x^2$, and higher powers of x can be neglected.

## EXPECTED BACKGROUND KNOWLEDGE

- Number System
- Four fundamental operations on numbers and expressions.
- Algebraic expressions and their simplifications.
- Indices and exponents.

## 8.1 WHAT IS A STATEMENT ?

In your daily interactions, you must have made several assertions in the form of sentences. Of these assertions, the ones that are **either true** or **false** are called **statement** or **propositions**. For instance,

"I am 20 years old" and "If $x = 3$, then $x^2 = 9$" are statements, but 'When will you leave ?' And 'How wonderful!' are not statements.

Notice that a statement has to be a definite assertion which can be true or false, but not both. For example, '$x - 5 = 7$' is not a statement, because we don't know what $x$, is. If $x = 12$, it is true, but if $x = 5$, 'it is not true. Therefore, '$x - 5 = 7$' is not accepted by mathematicians as a statement.

But both '$x - 5 = 7 \Rightarrow x = 12$' and $x - 5 = 7$ for any real number $x$' are statements, the first one true and the second one false.

**Example 8.1** Which of the following sentences is a statement ?

(i) India has never had a woman President.

(ii) 5 is an even number.

(iii) $x^n > 1$

(iv) $(a + b)^2 = a^2 + 2ab + b^2$

**Solution :** (i) and (ii) are statements, (i) being true and (ii) being false. (iii) is not a statement, since we can not determine whether it is true or false, unless we know the range of values that $x$ and $y$ can take.

Now look at (iv). At first glance , you may say that it is not a statement, for the very same reasons that (iii) is not. But look at (iv) carefully. It is true for any value of $a$ and $b$. It is an identity. Therefore, in this case, even though we have not specified the range of values for $a$ and $b$, (iv) is a statement.

Some statements, like the one given below are about natural numbers in general. Let us look at the statement given below :

$$1+2+\ldots+n=\frac{n(n+1)}{2}$$

This involves a general natural number $n$. Let us call this statement P ($n$) [P stands for proposition].

Then P (1) would be $1=\frac{1(1+1)}{2}$

Similarly, P (2) would be the statement

$$1+2=\frac{2(2+1)}{2} \text{ and so on.}$$

Let us look at some examples to help you get used to this notation.

**Example 8.2** If P (n) denotes $2^n > n-1$, write P (1), P ($k$) and P ($k$+1), where $k \in N$ .

**Solution :** Replacing $n$ by 1, $k$ and $k$ + 1, respectively in P ($n$), we get

$P(1) : 2^1 > 2 - 1$, i.e., $2 > 1$

$P(k) : 2^k > k - 1$

$P(k + 1) : 2^{k+1} > (k + 1) - 1$, i.e., $2^{k+1} > k$

**Example 8.3** If P ($n$) is the statement

$$‘1 + 4 + 7 + (3n - 2) = \frac{n(3n-1)}{2}$$

write $P(1)$, $P(k)$ and $P(k + 1)$.

**Solution :** To write $P(1)$, the terms on the left hand side (LHS) of $P(n)$ continue till $3\times1-2$, i.e., 1. So, P (1) will have only one term in its LHS, i.e., the first term.

Also, the right hand side (RHS)of $P(1)=\frac{1\times(3\times1-1)}{2}=1$

Therefore, $P(1)$ is $1 = 1$.

Replacing $n$ by 2, we get

$$P(2) : 1 + 4 = \frac{2 \times (3 \times 2 - 1)}{2}, \text{ i.e., } 5 = 5.$$

Replacing $n$ by $k$ and $k + 1$, respectively, we get

$$P(k) : 1 + 4 + 7 + .... + (3k - 2) = \frac{k(3k-1)}{2}$$

$$P(k + 1) : 1 + 4 + 7 + .... + (3k - 2) + [3\,(k + 1) - 2]$$

$$= \frac{(k+1)[3(k+1)-1]}{2}$$

$$\text{i.e. , } 1 + 4 + 7 +.... + (3k + 1) = \frac{(k+1)[(3k+2)}{2}$$

## CHECK YOUR PROGRESS 8.1

1. Determine which of the following are statements :

   (a) $1 + 2 + 4 \quad ........+ 2^n > 20$ (b) $1 + 2 + 3 + ....... + 10 = 99$

   (c) Chennai is much nicer than Mumbai. (d) Where is Timbuktu ?

   (e) $\frac{1}{1\times 2} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for $n = 5$ (f) $\operatorname{cosec}\theta < 1$

2. Given that $P(n)$ : 6 is a factor of $n^3 + 5n$, write $P(1)$, $P(2)$, $P(k)$ and $P(k+1)$ where $k$ is a natural number.

3. Write $P(1)$, $P(k)$ and $P(k + 1)$, if $P(n)$ is:

   (a) $2^n \geq n + 1$ (b) $(1 + x)^n \geq 1 + nx$

   (c) $n\,(n +1)\,(n + 2)$ is divisible by 6. (d) $(x^n - y^n)$ is divisible by $(x - y)$.

   (e) $(ab)^n = a^n b^n$ (f) $\left(\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}\right)$ is a natural number.

4. Write $P(1)$, $P(2)$, $P(k)$ and $P(k +1)$, if $P(n)$ is :

   (a) $\frac{1}{1\times 2} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$

   (b) $1 + 3 + 5 + ........ + (2n - 1) = n^2$

Notes

(c) $(1\times 2)+(2\times 3)+\ldots.+n(n+1) < n(n+1)^2$

(d) $\dfrac{1}{1\times 3}+\dfrac{1}{3\times 5}+\ldots\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$

Now ,when you are given a statement like the ones given in Examples 2 and 3, how would you check whether it is true or false ? One effective method is mathematical induction, which we shall now discuss.

## 8.2 THE PRINCIPLE OF MATHEMATICAL INDUCTION

In your daily life, you must be using various kinds of reasoning depending on the situation you are faced with. For instance, if you are told that your friend just had a child, you would know that it is either a girl or a boy. In this case, you would be applying general principles to a particular case. This form of reasoning is an example of **deductive** logic.

Now let us consider another situation. When you look around, you find students who study regularly, do well in examinations, you may formulate the general rule (rightly or wrongly) that "any one who studies regularly will do well in examinations". In this case, you would be formulating a general principle (or rule) based on several particular instances. Such reasoning is inductive, a process of reasoning by which general rules are discovered by the observation and consideration of several individual cases. Such reasoning is used in all the sciences, as well as in Mathematics.

Mathematical induction is a more precise form of this process. This precision is required because a statement is accepted to be true mathematically only if it can be shown to be true for each and every case that it refers to. The following principle allows us to check if this happens.

**The Principle of Mathematical Induction:**

Let $P(n)$ be a statement involving a natural number $n$. If

(i) it is true for $n = 1$, i.e., $P(1)$ is true; and

(ii) assuming $k \geq 1$ and $P(k)$ to be true, it can be proved that $P(k+1)$ is true; then $P(n)$ must be true for every natural number $n$.

***Note** that condition (ii) above **does not** say that P(k) **is** true. It says that **whenever** P(k) is true, then P( k + 1) is true'.*

Let us see, for example, how the principle of mathematical induction allows us to conclude that $P(n)$ is true for $n = 11$.

By (i) $P(1)$ is true. As $P(1)$ is true, we can put $k = 1$ in (ii), So $P(1 + 1)$, i.e., $P(2)$ is true. As $P(2)$ is true, we can put $k = 2$ in (ii) and conclude that $P(2 + 1)$, i.e., $P(3)$ is true. Now put $k = 3$ in (ii), so we get that $P(4)$ is true. It is now clear that if we continue like this, we shall get that $P(11)$ is true.

It is also clear that in the above argument, 11 does not play any special role. We can prove that $P(137)$ is true in the same way. Indeed, it is clear that $P(n)$ is true for all $n > 1$.

Let us now see, through examples, how we can apply the principle of mathematical induction to prove various types of mathematical statements.

**Example 8.4** Prove that

$$1+2+3+\cdots+n=\frac{n}{2}(n+1)\text{, where } n \text{ is a natural number.}$$

**Solution:** We have

$$P(n) : 1 + 2 + 3 + ... + n = \frac{n}{2}(n + 1)$$

Therefore, $P(1)$ is '$1 = \frac{1}{2}(1+1)$', which is true,.

Therefore, $P(1)$ is true.

Let us now see, if $P(k + 1)$ is true whenever $P(k)$ is true.

Let us, therefore, assume that $P(k)$ is true, i.e.,

$$1 + 2 + 3 ... + k = \frac{k}{2}(k+1) \qquad ....(i)$$

Now, $P(k + 1)$ is $1 + 2 + 3 + ... + k + (k + 1) = \dfrac{(k+1)(k+2)}{2}$

It will be true, if we can show that LHS = RHS

The LHS of $P(k + 1) = (1 + 2 + 3 ... + k) + (k + 1)$

$$= \frac{k}{2}(k+1) + (k + 1) \qquad ....[\text{From (i)}]$$

$$= (k + 1)\left(\frac{k}{2}+1\right)$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \text{RHS of } P(k+1)$$

So, $P(k + 1)$ is true, if we assume that $P(k)$ is true.

Since $P(1)$ is also true, both the conditions of the principle of mathematical induction are fulfilled, we conclude that the given statement is true for every natural number $n$.

As you can see, we have proved the result in three steps – the **basic step** [i.e., checking (i)], the **Induction step** [i.e., checking (ii)], and hence arriving at the end result.

**Example 8.5** Prove that

$$1.2 + 2.2^2 + 3.2^3 + 4.2^4 + \ldots + n.2^n = (n-1).2^{n+1} + 2,$$

where $n$ is a natural number.

**Solution :** Here $P(n)$ is $1.2^1 + 2.2^2 + 3.2^3 + \ldots + n.2^n = (n-1)\,2^{n+1} + 2$

Therefore, $P(1)$ is $1.2^1 = (1-1)2^{1+1} + 2$, i.e., $2 = 2$.

So, $P(1)$ is true.

We assume that $P(k)$ is true, i.e.,

$$1.2^1 + 2.2^2 + 3.2^3 + \ldots + k.2^k = (k-1)2^{k+1} + 2 \qquad \ldots.(i)$$

Now will prove that $P(k+1)$ is true.

Now $P(k+1)$ is

$$1.2^1 + 2.2^2 + 3.2^3 + \ldots + k.2^k + (k+1)2^{k+1} = [(k+1)-1]2^{(k+1+1)} + 2$$

$$= k.2^{k+2} + 2$$

LHS of $P(k+1) = (1.2^1 + 2.2^2 + 3.2^3 + \ldots. + k.2k + (k+1)\,2^{k+1}$

$$= 2^{k+1}\left[(k-1)+(k+1)\right]+2$$

$$= 2^{k+1}(2k)+2 \qquad \text{[Using (i)]}$$

$$= k2^{k+2}+2$$

$= \text{RHS of } P(k+1)$

Therefore, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number $n$.

**Example 8.6** For ever natural number $n$, prove that $\left(x^{2n-1}+y^{2n-1}\right)$ is divisible by $(x+y)$, where $x, y \in N$.

**Solution:** Let us see if we can apply the principle of induction here. Let us call $P(n)$ the statement

‘$\left(x^{2n-1}+y^{2n-1}\right)$ is divisible by $(x+y)$’,

Then $P(1)$ is ‘$\left(x^{2-1}+y^{2-1}\right)$ is divisible by $(x+y)$’, i.e., ‘$(x+y)$ is divisible by $(x+y)$’, which is true.

Therefore, $P(1)$ is true.

Let us now assume that $P(k)$ is true for some natural number $k$, i.e., $(x^{2k-1} + y^{2k-1})$ is divisible by $(x + y)$.

This means that for some natural number $t$, $x^{2k-1} + y^{2k-1} = (x+y)t$

Then, $x^{2k-1} = (x+y)t - y^{2k-1}$

We wish to prove that P $(k+1)$ is true, i.e., '$[x^{2(k+1)-1} + y^{2(k+1)-1}]$ is divisible by $(x+y)$' is true.

Now,

$$\begin{aligned}
x^{2(k+1)-1} + y^{2(k+1)-1} &= x^{2k+1} + y^{2k+1} \\
&= x^{2k-1+2} + y^{2k+1} \\
&= x^{2}\cdot x^{2k-1} + y^{2k+1} \\
&= x^{2}\cdot[(x+y)t - y^{2k-1}] + y^{2k+1} \qquad \text{[From (1)]} \\
&= x^{2}(x+y)t - x^{2}y^{2k-1} + y^{2k+1} \\
&= x^{2}(x+y)t - x^{2}y^{2k-1} + y^{2}y^{2k-1} \\
&= x^{2}(x+y)t - y^{2k-1}(x^{2} - y^{2}) \\
&= (x+y)[x^{2}t - (x-y)\,y^{2k-1}]
\end{aligned}$$

which is divisible by of $(x + y)$.

Thus, $P\,(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number $n$.

**Example 8.7** Prove that $2^n > n$ for every natural number $n$.

**Solution:** We have $P(n) : 2^n > n$.

Therefore, $P(1) : 2^1 > 1$, i.e., $2 > 1$, which is true.

We assume $P(k)$ to be true, that is,

$2^k > k$ ... (i)

We wish to prove that $P(k + 1)$ is true, i.e. $2^{k+1} > k + 1$.

Now, multiplying both sides of (i) by 2, we get

$2^{k+1} > 2k$

$\Rightarrow 2^{k+1} > k + 1$, since $k > 1$.

Therefore, $P(k + 1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number $n$.

Sometimes, we need to prove a statement for all natural numbers greater than a particular natural number, say $a$ (as in Example 8.8 below). In such a situation, we replace $P(1)$ by $P(a + 1)$ in the statement of the principle.

**Example 8.8** Prove that

$n^2 > 2(n + 1)$ for all $n \geq 3$, where $n$ is a natural number.

**Solution:** For $n \geq 3$, let us call the following statement

$$P(n) : n^2 > 2\ (n + 1)$$

Since we have to prove the given statement for $n \geq 3$, the first relevant statement is $P(3)$. We, therefore, see whether $P(3)$ is true.

$P(3) : 3^2 > 2 \times 4$, i.e. $9 > 8$.

So, $P$ (3) is true.

Let us assume that $P(k)$ is true, where $k \geq 3$, that is

$$k^2 > 2(k + 1) \qquad \text{........ (i)}$$

We wish to prove that $P$ $(k + 1)$ is true.

$$\text{P}\ (k + 1) : (k + 1)^2 > 2(k + 2)$$

$$\begin{aligned}\text{LHS of } P\ (k + 1) &= (k + 1)^2\\ &= k^2 + 2k + 1\\ &> 2\ (k + 1) + 2k + 1 \qquad \text{... [ By (i) ]}\\ &> 3 + 2k + 1, \text{ since } 2\ (k + 1) > 3.\\ &= 2\ (k + 2),\end{aligned}$$

Thus, $(k + 1)^2 > 2(k + 2)$

Therefore, $P(k + 1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number $n \geq 3$.

**Example 8.9** Using principle of mathematical induction, prove that

$$\left(\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}\right) \text{ is a natural number for all natural numbers } n.$$

**Solution :**

Let $P(n) : \left(\dfrac{n^5}{5} + \dfrac{n^3}{3} + \dfrac{7n}{15}\right)$ be a natural number.

$\therefore P(1) : \left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$ is a natural number.

or, $\frac{1}{5}+\frac{1}{3}+\frac{7}{15}=\frac{3+5+7}{15}=\frac{15}{15}=1$, which is a natural number.

$\therefore$ $P(1)$ is true.

Let $P(k) : \left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)$ is a natural number be true ... (i)

Now $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$

$$=\frac{1}{5}\left[k^5+5k^4+10k^3+10k^2+5k+1\right]+\frac{1}{3}\left[k^3+3k^2+3k+1\right]+\left(\frac{7}{15}k+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+3k^2+2k\right)+\left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+3k^2+2k\right)+1 \qquad \text{...(ii)}$$

By (i), $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

also $k^4+2k^3+3k^2+2k$ is a natural number and 1 is also a natural number.

$\therefore$ (ii) being sum of natural numbers is a natural number.

$\therefore$ $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(n)$ is true for all natural numbers $n$.

Hence, $\left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$ is a natural number for all natural numbers $n$.

## CHECK YOUR PROGRESS 8.2

1. Using the principle of mathematical induction, prove that the following statements hold for any natural number $n$:

   (a) $1^2+2^2+3^2+........+n^2=\frac{n}{6}(n+1)(2n+1)$

   (b) $1^3+2^3+3^3+........+n^3=(1+2+.....+n)^2$

   (c) $1 + 3 + 5 + ........ + (2n-1) = n^2$

   (d) $1+4+7+........+(3n-2)=\frac{n}{2}(3n-1)$

2. Using principle of mathematical induction, prove the following equalities for any natural number $n$:

   (a) $\frac{1}{1\times2}+\frac{1}{2\times3}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}$

   (b) $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

   (c) $(1\times2)+(2\times3)+....+n(n+1)=\frac{n(n+1)(n+2)}{3}$

3. For every natural number $n$, prove that

   (a) $n^3+5n$ is divisible by 6. (b) $(x^n-1)$ is divisible by $(x-1)$.

   (c) $(n^3+2n)$ is divisible by 3. (d) 4 divides $(n^4+2n^3+n^2)$.

4. Prove the following inequalities for any natural number $n$:

   (a) $3^n \geq 2n+1$ (b) $4^{2n} > 15n$

   (c) $1+2+....+n<\frac{1}{8}(2n+1)^2$

5. Prove the following statements using induction:

   (a) $2^n > n^2$ for $n \geq 5$, where $n$ *is any natural number.*

   (b) $\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{2n}>\frac{13}{24}$ for any natural number $n$ greater than 1.

6. Prove that $n(n^2-1)$ is divisible by 3 for every natural number $n$ greater than 1.

To prove that a statement $P(n)$ is true for every $n \in N$, both the basic as well as the induction steps must hold.

If even one of these conditions does not hold, then the proof is invalid. For instance, if $P(n)$ is' $(a+b)^n \leq a^n+b^n$' for all reals $a$ and $b$, then $P(1)$ is certainly true. But, $P(k)$ being true does not imply the truth of $P(k + 1)$. So, the statement is not true for every natural number $n$. (For instance, $(2+3)^2 \nleq 2^2 + 3^2$).

As another example, take $P(n)$ to be $n > \frac{n}{2} + 20$.

In this case, $P(1)$ is not true. But the induction step is true. Since $P(k)$ being true.

$$\Rightarrow k > \frac{k}{2} + 20$$

$$\Rightarrow k+1 > \frac{k}{2} + 20 + 1 > \frac{k}{2} + 20 + \frac{1}{2} = \frac{k+1}{2} + 20$$

$$\Rightarrow P(k+1) \text{ is true.}$$

Or if we want a statement which is false for all $n$, then take $P(n)$ to be $n < \frac{n}{2} + 20$.

And, as you can see, $P(n)$ is false for large values of $n$ say $n = 100$.

## 8.3 THE BINOMIAL THEOREM FOR A NATURAL EXPONENT

You must have multiplied a binomial by itself, or by another binomial. Let us use this knowledge to do some expansions. Consider the binomial $(x + y)$. Now,

$$(x+y)^1 = x + y$$

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x+y)^3 = (x+y)(x+y)^2 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = (x+y)(x+y)^3 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^5 = (x+y)(x+y)^4 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

and so on.

In each of the equations above, the right hand side is called the binomial expansion of the left hand side.

Note that in each of the above expansions, we have written the power of a binomial in the expanded form in such a way that the terms are in descending powers of the first term of the binomial (which is $x$ in the above examples). If you look closely at these expansions, you would also observe the following:

1. The number of terms in the expansion is one more than the exponent of the binomial. For example, in the expansion of $(x + y)^4$, the number of terms is 5.

2. The exponent of $x$ in the first term is the same as the exponent of the binomial, and the exponent decreases by 1 in each successive term of the expansion.

3. The exponent of $y$ in the first term is zero (as $y^0 = 1$). The exponent of $y$ in the second term is 1, and it increases by 1 in each successive term till it becomes the exponent of the binomial in the last term of the expansion.

4. The sum of the exponents of $x$ and $y$ in each term is equal to the exponent of the binomial. For example, in the expansion of $(x + y)^5$, the sum of the exponents of $x$ and $y$ in each term is 5.

If we use the combinatorial co-efficients, we can write the expansion as

$$(x+y)^3 = {}^3C_0\,x^3 + {}^3C_1x^2y + {}^3C_2x\,y^2 + {}^3C_3\,y^3$$

$$(x+y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4\,y^4$$

$$(x+y)^5 = {}^5C_0x^5 + {}^5C_1x^4y + {}^5C_2x^3y^2 + {}^5C_3x^2y^3 + {}^5C_4xy^4 + {}^5C_5\,y^5,$$

and so on.

More generally, we can write the binomial expansion of $(x + y)^n$, where $n$ is a positive integer, as given in the following theorem. This statement is called the **binomial theorem for a natural** (or **positive integral**) **exponent**.

**Theorem 8.1**

$$(x+y)^n = {}^nC_0\,x^n + {}^nC_1x^{n-1}y^1 + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_{n-1}\,xy^{n-1} + {}^nC_ny^n \quad \ldots(A)$$

where $n \in N$ and $x, y \in R$.

**Proof :** Let us try to prove this theorem, using the principle of mathematical induction.

Let statement (A) be denoted by $P(n)$, i.e.,

$$P(n):\ (x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + {}^nC_3x^{n-3}y^3 + \ldots + {}^nC_{n-1}\,xy^{n-1} + {}^nC_ny^n \quad \ldots(i)$$

Let us examine whether $P(1)$ is true or not.

From ($i$), we have

$$P(1) : (x+y)^1 = {}^1C_0x + {}^1C_1\,y = 1\times x + 1\times y$$

i.e., $(x + y)^1 = x + y$

Thus, $P(1)$ holds.

Now, let us assume that $P(k)$ is true, i.e.,

$$P(k):\ (x+y)^k = {}^kC_0x^k + {}^kC_1x^{k-1}y + {}^kC_2x^{k-2}y^2 + {}^kC_3x^{k-3}y^3 + \ldots + {}^kC_{k-1}xy^{k-1} + {}^kC_k\,y^k \quad \ldots(ii)$$

Assuming that $P(k)$ is true, if we prove that $P(k + 1)$ is true, then $P(n)$ holds, for all $n$. Now,

$$(x+y)^{k+1} = (x+y)(x+y)^k = (x+y)\left({}^kC_0x^k + {}^kC_1x^{k-1}y + {}^kC_2x^{k-2}y^2 + \ldots + {}^kC_{k-1}xy^{k-1} + {}^kC_ky^k\right)$$

$$= {}^kC_0x^{k+1} + {}^kC_0x^ky + {}^kC_1x^ky + {}^kC_1x^{k-1}y^2 + {}^kC_2x^{k-1}y^2 + {}^kC_2x^{k-2}y^3 +$$

$$\ldots + {}^kC_{k-1}x^2y^{k-1} + {}^kC_{k-1}xy^k + {}^kC_kxy^k + {}^kC_ky^{k+1}$$

i.e. $(x+y)^{k+1} = {}^kC_0x^{k+1} + ({}^kC_0 + {}^kC_1)x^ky + ({}^kC_1 + {}^kC_2)x^{k-1}y^2 +$

$$\ldots + ({}^kC_{k-1} + {}^kC_k)xy^k + {}^kC_ky^{k+1} \qquad \ldots(iii)$$

From Lesson 7, you know that ${}^kC_0 = 1 = {}^{k+1}C_0$ ....(*iv*)

and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$

Also, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$

Therefore, ${}^kC_0 + {}^kC_1 = {}^{k+1}C_1$ .....(*v*)

$${}^kC_1 + {}^kC_2 = {}^{k+1}C_2$$

$${}^kC_2 + {}^kC_3 = {}^{k+1}C_3$$

........................

........................ and so on

Using (iv) and (v), we can write (iii) as

$$(x+y)^{k+1} = {}^{k+1}C_0x^{k+1} + {}^{k+1}C_1x^ky + {}^{k+1}C_2x^{k-1}y^2 +$$

$$\ldots + {}^{k+1}C_kxy^k + {}^{k+1}C_{k+1}y^{k+1}$$

which shows that P($k$ +1) is true.

Thus, we have shown that (a) $P(1)$ is true, and (b) if $P(k)$ is true, then $P(k+1)$ is also true.

Therefore, by the principle of mathematical induction, $P(n)$ holds for any value of $n$. So, we have proved the binomial theorem for any natural exponent.

This result is supported to have been proved first by the famous Arab poet Omar Khayyam, though no one has been able to trace his proof so far.

We will now take some examples to illustrate the theorem.

**Example 8.10** Write the binomial expansion of $(x + 3y)^5$.

**Solution :** Here the first term in the binomial is $x$ and the second term is $3y$. Using the binomial theorem, we have

$$(x + 3y)^5 = {}^5C_0x^5 + {}^5C_1x^4(3y)^1 + {}^5C_2x^3(3y)^2 + {}^5C_3x^2(3y)^3 + {}^5C_4x(3y)^4 + {}^5C_5(3y)^5$$

$$= 1\times x^5 + 5x^4\times 3y + 10x^3\times(9y^2) + 10x^2\times(27y^3) + 5x\times(81y^4) + 1\times 243y^5$$

$$= x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$$

Thus, $(x+3y)^5 = x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$

**Example 8.11** Expand $(1+a)^n$ in terms of powers of $a$, where $a$ is a real number.

**Solution :** Taking $x = 1$ and $y = a$ in the statement of the binomial theorem, we have

$$(1+a)^n = {}^nC_0(1)^n + {}^nC_1(1)^{n-1}a + {}^nC_2(1)^{n-2}a^2 + \ldots + {}^nC_{n-1}(1)a^{n-1} + {}^nC_na^n$$

i.e., $$(1+a)^n = 1 + {}^nC_1a + {}^nC_2a^2 + \ldots + {}^nC_{n-1}a^{n-1} + {}^nC_na^n \qquad \ldots \text{(B)}$$

(B) is another form of the statement of the binomial theorem.

The theorem can also be used in obtaining the expansions of expressions of the type

$$\left(x+\frac{1}{x}\right)^5, \left(\frac{y}{x}+\frac{1}{y}\right)^5, \left(\frac{a}{4}+\frac{2}{a}\right)^5, \left(\frac{2t}{3}-\frac{3}{2t}\right)^6, \text{ etc.}$$

Let us illustrate it through an example.

**Example 8.12** Write the expansion of $\left(\frac{y}{x}+\frac{1}{y}\right)^4$, where $x, y \neq 0$.

**Solution :** We have :

$$\left(\frac{y}{x}+\frac{1}{y}\right)^4 = {}^4C_0\left(\frac{y}{x}\right)^4 + {}^4C_1\left(\frac{y}{x}\right)^3\left(\frac{1}{y}\right) + {}^4C_2\left(\frac{y}{x}\right)^2\left(\frac{1}{y}\right)^2$$

$$+ {}^4C_3\left(\frac{y}{x}\right)\left(\frac{1}{y}\right)^3 + {}^4C_4\left(\frac{1}{y}\right)^4$$

$$= 1\times\frac{y^4}{x^4} + 4\times\frac{y^3}{x^3}\times\frac{1}{y} + 6\times\frac{y^2}{x^2}\times\frac{1}{y^2} + 4\times\left(\frac{y}{x}\right)\times\frac{1}{y^3} + 1\times\frac{1}{y^4}$$

$$= \frac{y^4}{x^4} + 4\frac{y^2}{x^3} + \frac{6}{x^2} + \frac{4}{xy^2} + \frac{1}{y^4}$$

**Example 8.13** The population of a city grows at the annual rate of 3%. What percentage increase is expected in 5 years ? Give the answer up to 2 decimal places.

**Solution :** Suppose the population is $a$ at present. After 1 year it will be

$$a+\frac{3}{100}a=a\left(1+\frac{3}{100}\right)$$

After 2 years, it will be $a\left(1+\frac{3}{100}\right)+\frac{3}{100}\left[a\left(1+\frac{3}{100}\right)\right]$

$$= a\left(1+\frac{3}{100}\right)\left(1+\frac{3}{100}\right) = a\left(1+\frac{3}{100}\right)^2$$

Similarly, after 5 years, it will be $a\left(1+\frac{3}{100}\right)^5$

Using the binomial theorem, and ignoring terms involving more than 3 decimal places, we get

$$a\left(1+\frac{3}{100}\right)^5 \approx a\left[1+5(0.03)+10(0.03)^2\right] = a\times 1.159$$

So, the increase is $0.159\times 100\% = \frac{159}{1000}\times 100\times\frac{1}{100} = 15.9\%$ in 5 years.

**Example 8.14** Using binomial theorem, evaluate

(*i*) $102^4$ (*ii*) $97^3$

**Solution :**

(*i*) $102^4$ $= (100 + 2)^4$

$= {}^4C_0\,(100)^4 + {}^4C_1(100)^3.2 + {}^4C_2(100)^2.2^2 + {}^4C_3\,(100).2^3 + {}^4C_4\,.2^4$

$= 100000000 + 8000000 + 240000 + 3200 + 16$

$= 108243216$

(*ii*) $(97)^3$ $= (100-3)^3$

$= {}^3C_0(100)^3 - {}^3C_1(100)^2.3 + {}^3C_2(100).3^2 - {}^3C_3.3^3$

$= 1000000 - 90000 + 2700 - 27$

$= 1002700 - 90027$

$= 912673$

## CHECK YOUR PROGRESS 8.3

1. Write the expansion of each of the following :

   (a) $(2a + b)^3$ (b) $(x^2 - 3y)^6$ (c) $(4a - 5b)^4$ (d) $(ax + by)^n$

2. Write the expansions of :

   (a) $(1 - x)^7$ (b) $\left(1+\frac{x}{y}\right)^7$ (c) $(1 + 2x)^5$

3. Write the expansions of :

   (a) $\left(\frac{a}{3}+\frac{b}{2}\right)^5$ (b) $\left(3x-\frac{5}{x^2}\right)^7$ (c) $\left(x+\frac{1}{x}\right)^4$ (d) $\left(\frac{x}{y}+\frac{y}{x}\right)^5$

4. Suppose I invest Rs. 1 lakh at 18% per year compound interest. What sum will I get back after 10 years? Give your answer up to 2 decimal places.

5. The population of bacteria increases at the rate of 2% per hour. If the count of bacteria at 9 a.m. is $1.5\times10^5$, find the number at 1 p.m. on the same day.

6. Using binomial theorem, evaluate each of the following :

   (*i*) $(101)^4$ (*ii*) $(99)^4$ (*iii*) $(1.02)^3$ (iv) $(0.98)^3$

## 8.4 GENERAL AND MIDDLE TERMS IN A BINOMIAL EXPANSION

Let us examine various terms in the expansion (A) of $(x + y)^n$, i.e., in

$$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + {}^nC_3x^{n-3}y^3 + ... + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$$

We observe that

the first term is ${}^nC_0x^n$, i.e., ${}^nC_{1-1}x^ny^0$;

the second term is ${}^nC_1x^{n-1}y$, i.e., ${}^nC_{2-1}x^{n-1}y^1$;

the third term is ${}^nC_2x^{n-2}y^2$, i.e., ${}^nC_{3-1}x^{n-2}y^2$;

and so on.

From the above, we can generalise that

the $(r + 1)^{th}$ term is ${}^nC_{(r+1)-1}x^{n-r}y^r$, i.e., ${}^nC_rx^{n-r}y^r$.

If we denote this term by $T_{r+1}$, we have

$$T_{r+1} = {}^nC_r x^{n-r} y^r$$

$T_{r+1}$ is generally referred to as the **general term** of the binomial expansion.

Let us now consider some examples and find the general terms of some expansions.

**Example 8.15** Find the $(r + 1)^{\text{th}}$ term in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$, where $n$ is a natural number. Verify your answer for the first term of the expansion.

**Solution :** The general term of the expansion is given by :

$$T_{r+1} = {}^nC_r (x^2)^{(n-r)} \left(\frac{1}{x}\right)^r$$

$$= {}^nC_r x^{2n-2r} \frac{1}{x^r}$$

$$= {}^nC_r x^{2n-3r} \qquad \ldots(i)$$

Hence, the $(r + 1)$th term in the expansion is ${}^nC_r x^{2n-3r}$.

On expanding $\left(x^2 + \frac{1}{x}\right)^n$, we note that the first term is $(x^2)^n$ or $x^{2n}$.

Using $(i)$, we find the first term by putting $r = 0$.

Since $T_1 = T_{0+1}$

$$\therefore \qquad T_1 = {}^nC_0 x^{2n-0} = x^{2n}$$

This verifies that the expression for $T_{r+1}$ is correct for $r + 1 = 1$.

**Example 8.16** Find the fifth term in the expansion of

$$\left(1 - \frac{2}{3}x^3\right)^6$$

**Solution :** Using here $T_{r+1} = T_5$ which gives $r + 1 = 5$, i.e., $r = 4$.

Also $n = 6$ and let $a = \frac{-2}{3}x^3$.

$$T_5 = {}^6C_4 \left(-\frac{2}{3}x^3\right)^4$$

$$= {}^6C_2\left(\frac{16}{81}x^{12}\right)$$

$$= \frac{6\times5}{2}\times\frac{16}{81}\times x^{12} = \frac{80}{27}x^{12}$$

Thus, the fifth term in the expansion is $\frac{80}{27}x^{12}$.

## CHECK YOUR PROGRESS 8.4

1. For a natural number $n$, write the $(r + 1)^{th}$ term in the expansion of each of the following:

(a) $(2x+y)^n$ (b) $(2a^2-1)^n$

(c) $(1-a)^n$ (d) $\left(3+\frac{1}{x^2}\right)^n$

2. Find the specified terms in each of the following expansions:

(a) $(1+2y)^8$; 6th term (b) $(2x+3)^7$; 4th term

(c) $(2a-b)^{11}$; 7th term (d) $\left(x+\frac{1}{x}\right)^6$; 4th term

(e) $\left(x^3-\frac{1}{x^2}\right)^7$; 5th term

Now that you are familiar with the general term of an expansion, let us see how we can obtain the **middle term** (or terms) of a binomial expansion. Recall that the number of terms in a binomial expansion is always one more than the exponent of the binomial. This implies that if the exponent is even, the number of terms is odd, and if the exponent is odd, the number of terms is even. Thus, while finding the middle term in a binomial expansion, we come across two cases:

**Case 1 :** When $n$ is even.

To study such a situation, let us look at a particular value of $n$, say $n = 6$. Then the number of terms in the expansion will be 7. From Fig. 8.1, you can see that there are three terms on either side of the fourth term.

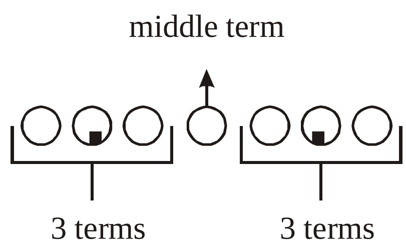


Fig. 8.1

In general, when the exponent $n$ of the binomial is even, there are $\frac{n}{2}$ terms on either side of the $\left(\frac{n}{2}+1\right)$th term. Therefore, the $\left(\frac{n}{2}+1\right)$th term is the middle term.

**Case 2:** When $n$ is odd

Let us take $n$ = 7 as an example to see what happens in this case. The number of terms in the expansion will be 8. Looking at Fig. 8.2, do you find any one middle term in it? There is not. But we can partition the terms into two equal parts by a line as shown in the figure. We call the terms on either side of the partitioning line taken together, the middle terms. This is because there are an equal number of terms on either side of the two, taken together.

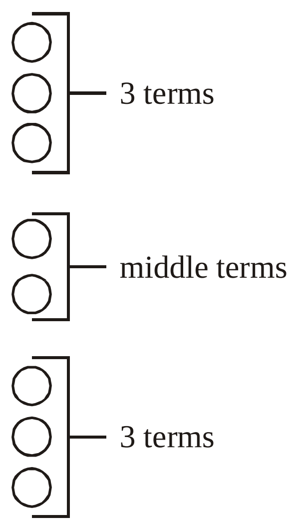


Fig. 8.2

Thus, in this case, there are two middle terms, namely, the fourth,

i.e., $\left(\frac{7+1}{2}\right)$ and the fifth, i.e., $\left(\frac{7+3}{2}\right)$ terms

Similarly, if $n = 13$, then the $\left(\frac{13+1}{2}\right)$th and the $\left(\frac{13+3}{2}\right)$th terms, i.e., the 7th and 8th terms are two middle terms, as is evident from Fig. 8.3.

From the above, we conclude that

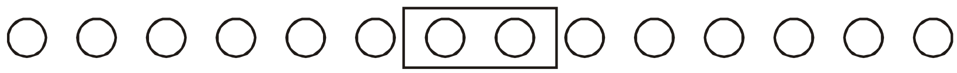

Fig. 8.3

**When the exponent *n* of a binomial is an odd natural number, then the $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms are two middle terms in the corresponding binomial expansion.**

Let us now consider some examples.

**Example 8.17** Find the middle term in the expansion of $(x^2+y^2)^8$.

**Solutuion :** Here $n = 8$ (an even number).

Therefore, the $\left(\frac{8}{2}+1\right)$th, i.e., the 5th term is the middle term.

Putting $r = 4$ in the general term $T_{r+1} = {}^8C_r(x^2)^{8-r}y^r$,

$$T_5 = {}^8C_4(x^2)^{8-4}(y^2)^4 = 70x^8y^8$$

**Example 8.18** Find the middle term(s) in the expansion of $\left(2x^2+\frac{1}{x}\right)^9$.

**Solution :** Here $n = 9$ (an odd number). Therefore, the $\left(\frac{9+1}{2}\right)th$ and $\left(\frac{9+3}{2}\right)th$ are middle terms. i.e. $T_5$ and $T_6$ are middle terms.

For finding $T_5$ and $T_6$, putting $r = 4$ and $r = 5$ in the general term

$$T_{r+1} = {}^9C_r(2x^2)^{9-r}\left(\frac{1}{x}\right)^6,$$

$$T_5 = {}^9C_4(2x^2)^{9-4}\left(\frac{1}{x}\right)^4$$

$$= \frac{9\times8\times7\times6}{4\times3\times2}\times(32x^{10})\times\left(\frac{1}{x}\right)^4 = 4032\ x^6$$

and $T_6 = 9_5\ (2\ x^2)^{9-5}\left(\frac{1}{x}\right)^5 = 2016\ x^3$

Thus, the two middle terms are $4032\ x^6$ and $2016\ x^3$.

## CHECK YOUR PROGRESS 8.5

1. Find the middle term(s) in the expansion of each of the following :

(a) $(2x+y)^{10}$ (b) $\left(1+\frac{2}{3}x^3\right)^8$

(c) $\left(x+\frac{1}{x}\right)^6$ (d) $(1-x^2)^{10}$

2. Find the middle term(s) in the expansion of each of the following :

(a) $(a+b)^7$ (b) $(2a-b)^9$

(c) $\left(\frac{3x}{4}-\frac{4y}{3}\right)^7$ (d) $\left(x+\frac{1}{x^2}\right)^{11}$

## 8.5 BINOMIAL THEOREM FOR RATIONAL EXPONENTS

So far you have applied the binomial theorem only when the binomial has been raised to a power which is a natural number. What happens if the exponent is a negative integer, or if it is a fraction? We will state the result that allows us to still have a binomial expansion, but it will have infinite terms in this case.

The result is a generalised version of the earlier binomial theorem which you have studied.

**Theorem 8.2** **The Binomial Theorem for a Rational Exponent.**

If $r$ is a rational number, and $x$ is a real number such that $|x| < 1$, *then*

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots \qquad \text{... (D)}$$

We will not prove this result here, as it is beyond the scope of this course. In fact, even Sir Issac Newton, who is credited with stating this generalisation, stated it without proof in two letters, written in A.D. 1676. The proof was developed later, by other mathematicians, in stages. Among those who contributed to the proof of this theorem were English mathematician Colin Maclaurin (A.D. 1698-1746) for rational values of $r$, Giovanni Francesco, M.M. Salvemini (A.D. 1708-1783) and the German mathematician Abraham G. Kasther (A.D. 1719 - 1800) for integral values of $r$, the Swiss mathematician Leonhard Euler (AD 1707-1783) for fractional exponents and the Norwegian mathematician Neils Henrik Abel (1802-1829) for complex exponents. Let us consider some examples to illustrate the theorem.

**Example 8.19** Write the expansion of $(1 + x)^{-1}$, when $|x| < 1$.

**Solution :** Here $r = -1$ [with reference to (D) above].

Therefore,

$$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + .....$$

i.e., $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + ....$

Similarly, you can write the expansion $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + ....$

Note the above expansions. In case of $(1 + x)^{-1}$ all the terms have positive and negative signs alternate, while in the case of $(1 - x)^{-1}$ all the terms have positive sign.

You may have also observed the following points about the binomial expansion (D) in general;

1. If $r$ is a natural number, then (C) and (D) coincide for the case $|x| < 1$.

2. Note that ${}^rC_0 = 1,\ {}^rC_1 = r,\ {}^rC_2 = \frac{r(r-1)}{2!}$ etc. Thus, the coefficients $1,\ r, \frac{r(r-1)}{2!}$.... in (D) look like combinatorial coefficients.,

   However, recall that ${}^rC_s$ is defined for natural numbers $r$ and whole number $s$ only. Therefore,

   ${}^rC_0, {}^rC_1, {}^rC_2,$ etc. have no meaning in the present context.

3. The expression (D) will have an infinite number of terms.

4. The sum of the series on the RHS of (D) may not be meaningful if $x > 1$.

   For example, if we put $x = 2$ in Example 1, we have

   $(1 + 2)^{-1} = 1 - 2 + 4 - 8 + 16 - 32 + ...$

   i.e., $\frac{1}{3} = (1-2) + (4-8) + (16-32) + ...$

   i.e., $\frac{1}{3} = -1 - 4 - 16 - ...,$

which is clearly false.

Therefore, for (D) to hold, it is necessary that $|x| < 1$.

Let us look at some more examples of this binomial expansion.

**Example 8.20** Expand $(x + y)^r$, where $r$ is a rational number and $\left|\frac{y}{x}\right| < 1$.

Hence expand $(3 + 5p)^{2/5}$, when $|p| < \frac{3}{5}$.

**Solution :** $(x+y)^r = x^r\left(1+\frac{y}{x}\right)^r$ .... (1)

Since it is given that $\left|\frac{y}{x}\right| < 1$, we have

$$\left(1+\frac{y}{x}\right)^r = 1 + r\left(\frac{y}{x}\right) + \frac{r(r-1)}{2!}\left(\frac{y}{x}\right)^2 + \frac{r(r-1)(r-2)}{3!}\left(\frac{y}{x}\right)^3 + \ldots$$

Therefore, from (1), we have

$$(x+y)^r = x^r\left[1 + r\left(\frac{y}{x}\right) + \frac{r(r-1)}{2!}\left(\frac{y}{x}\right)^2 + \frac{r(r-1)(r-2)}{3!}\left(\frac{y}{x}\right)^3 + \ldots\right]$$

i.e., $(x+y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \ldots$ (2)

Now, to solve the second part of the question, note that $|p| < \frac{3}{5}$.

If $\left|\frac{5p}{3}\right| < 1$, then putting $x = 3,\ y = 5p,\ r = \frac{2}{5}$ in (2), we get

$$(3+5p)^{2/5} = 3^{2/5} + \frac{2}{5}(3)^{\frac{2}{5}-1}(5p)^1 + \frac{\frac{2}{5}\left(\frac{2}{5}-1\right)}{2!}(3)^{\frac{2}{5}-2}(5p)^2 + \ldots$$



$$= \quad 3^{2/5} + (3)^{-3/5}(2p) + \frac{\frac{2}{5}\left(-\frac{3}{5}\right)}{2}(3)^{-8/5}25p^2 + \ldots$$

$$= \quad 3^{2/5} + 3^{-3/5}(2p) - 3^{-3/5}p^2 + \ldots$$

The result we have just obtained in Example 8.20 is another form of the binomial theorem for a **rational exponent.** Let us restate it formally.

If $r$ is a rational number and $\left|\frac{y}{x}\right| < 1,$

$$(x+y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \ldots\ldots \text{ (E)}$$

Note that you could have expanded $(x+y)^r$ differently if $\left|\frac{y}{x}\right| > 1$ were true. In this case, you would have had $\left|\frac{x}{y}\right| < 1,$ and $(x+y)^r = y^r\left(1+\frac{x}{y}\right)^r = y^r + ry^{r-1}.x + \ldots$

Consequently, we have the following result:

**For a rational number $r$, an expression like $(ax + by)^r$ can be expanded in two different ways, depending on whether**

$$\left|\frac{by}{ax}\right| < 1 \ \text{ or } \ \left|\frac{ax}{by}\right| < 1$$

**Example 8.21** Expand $(x + y)^{-5}$ when

(i) $\left|\frac{y}{x}\right| < 1$ and (ii) $\left|\frac{x}{y}\right| < 1$ .

**Solution :** (i) Since $\left|\frac{y}{x}\right| < 1$ , using (E) we have

$$(x+y)^{-5} = x^{-5} + (-5)x^{-5-1}y + \frac{(-5)(-6)}{2!}x^{-5-2}y^2 + \frac{(-5)(-6)(-7)}{3!}x^{-5-3}y^3 + \ldots.$$

$$= \frac{1}{x^5} - \frac{5y}{x^6} + \frac{15y^2}{x^7} - \frac{35y^3}{x^8} + \ldots$$

(ii) Since $\left|\frac{x}{y}\right| < 1$, we have to write $(x + y)^{-5}$ in the form $(y+x)^{-5}$.

Using (E), we can write

$$(y + x)^{-5} = y^{-5} + (-5)\, y^{-5-1}x$$

$$+ \frac{(-5)(-6)}{2!} y^{-5-2}x^2 + \frac{(-5)(-6)(-7)}{3!} y^{-5-3}x^3 + \ldots$$

$$= \frac{1}{y^5} - \frac{5x}{y^6} + \frac{15x^2}{y^7} - \frac{35x^3}{y^8} + \ldots.$$

Note that in (i), we have obtained the expansion in ascending powers of $y$ while in (ii), we have obtained the expansion in ascending powers of $x$.

## CHECK YOUR PROGRESS 8.6

1. Expand each of the following :

   (a) $(1 - p)^{-3}$ for $|p| < 1$ (b) $(1 + 3x)^{4/3}$ for $|x| < \frac{1}{3}$

   (c) $(1-5z)^{\frac{6}{5}}$, for $|z| < \frac{1}{5}$

2. Expand each of the following :

   (a) $(27-6x)^{\frac{-2}{3}}$, for $\left|\frac{2x}{9}\right| < 1$ (b) $(2a + x)^{-3}$, for $\left|\frac{x}{2a}\right| < 1$

   (c) $(2+3y)^{\frac{1}{7}}$, for $|y| > \frac{2}{3}$

3. (a) State the condition under which the expansion of $(x + 2y)^{-5}$ will be valid in

   ($i$) ascending powers of $x$.

   ($ii$) ascending powers of $y$.

   Also, write down the expansion in each case.

   (b) Expand, $(3 + 6y)^{\frac{-4}{3}}$, stating the range of values of $y$ for which the expansion is valid.

Notes

## 8.9 USE OF BINOMIAL THEOREM IN APPROXIMATIONS

As you have seen, the binomial expansions sometime have infinitely many terms. In such cases, for further calculations; an approximate value involving only the first few terms may be enough for us. Let us illustrate some situations in which we find the approximate values.

**Example 8.22** Find the cube root of 1.03 up to three decimal places.

**Solution :** We want to find $(1.03)^{\frac{1}{3}}$ up to three decimal places.

Now $(1.03)^{1/3} = (1+0.03)^{\frac{1}{3}}$

Since $|0.03| < 1$, from (E), we have

$$(1+0.03)^{1/3} = 1+\frac{1}{3}(0.03)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(0.03)^2+\ldots \qquad (i)$$

Now, we need to approximate the value up to three decimal places. Since a non-zero digit in the fourth decimal place may affect the digit in the third place in the process of rounding off, we need to consider those terms in the expansion which produce a non-zero digit in the first, second, third or fourth decimal place.

Therefore, we can take the sum of the first three terms in the Expansion (*i*), and ignore the rest.

$$\therefore \qquad (1.03)^{1/3} \approx 1+0.01+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(0.0009)$$

$$= 1 + 0.01 - 0.0001$$

$$= 1.0099$$

$$\approx 1.010, \text{ taking the value up to three decimal places.}$$

Now, the digit after the third decimal place is greater than 5, so we have increased the third decimal place by 1.

Thus, the cube root of 1.03, up to three decimal places, is 1.010.

**Example 8.23** Assuming $y$ to be so small that $y^2$ and higher powers of $y$ can be neglected, find the value of $(1-2y)^{2/3}(4+5y)^{\frac{-3}{2}}$.

**Solution :** Note that $y$ is very small. So, we can assume that $|y| < \frac{1}{2}$. Then, using the binomial theorem, we get

$$(1-2y)^{\frac{2}{3}} = 1+\frac{2}{3}(-2y)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}(-2y)^2+...$$

and

$$(4+5y)^{\frac{-3}{2}} = 4^{\frac{-3}{2}}+\left(-\frac{3}{2}\right)(4)^{\frac{-3}{2}-1}(5y)+\frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}(4)^{\frac{-3}{2}-2}.(5y)^2+...$$

Since we can neglect terms containing $y^2$ and higher powers of $y$, we have

$$(1-2y)^{\frac{2}{3}} \approx 1+\frac{2}{3}(-2y) = 1-\frac{4}{3}y, \text{ and}$$

$$(4+5y)^{\frac{-3}{2}} \approx (4)^{\frac{-3}{2}} - \frac{3}{2}(4)^{\frac{-5}{2}}(5y)$$

$$= \frac{1}{8}-\frac{15}{64}y$$

Thus, the given product is approximately

$$\left(1-\frac{4}{3}y\right)\left(\frac{1}{8}-\frac{15}{64}y\right) = \frac{1}{8}-\frac{1}{6}y-\frac{15}{64}y+\frac{5}{16}y^2$$

$$\approx \frac{1}{8}-\frac{77}{192}y$$ , again neglecting the term containing $y^2$.

So, $(1–2y)^{\frac{2}{3}}(4+5y)^{\frac{-3}{2}}$ is $\frac{1}{8}-\frac{77}{192}y$, if we neglect the terms involving $y^2$ and higher powers of $y$.

## CHECK YOUR PROGRESS 8.7

1. Find the value of each of the following up to three decimal places:

   (a) $(1.02)^2$ (b) $(1.01)^{-3}$ (c) $(0.97)^{-4}$

   (d) $\sqrt[3]{7.60}$ [Hint : $(7.60)^{1/3} = (8–0.4)^{1/3}$]

   (e) $\sqrt[4]{82}$ [Hint: $(82)^{\frac{1}{4}} = (81+1)^{\frac{1}{4}} = 3\left(1+\frac{1}{81}\right)^{\frac{1}{4}}$]

   (f) $(24)^{\frac{-1}{2}}$ [Hint : $(24)^{\frac{-1}{2}} = (25–1)^{\frac{-1}{2}}$ ]

2. Assuming z to be so small that $z^2$ and higher powers of z can be neglected, find the value of

   (a) $(3 + 2z)^{-5}$ (b) $(1 +3z)^{\frac{2}{3}}(1 –5z)^{-2}$

(c) $\dfrac{\sqrt{1+z}+(1-z)^{2/3}}{(1+z)+\sqrt{1+z}}$ $\left[\text{ Hint : LHS } \approx \dfrac{1+\frac{1}{2}z+1-\frac{2}{3}z}{(1+z)+(1+\frac{1}{2}z)}\right]$

(d) $\dfrac{(1-z)^{\frac{1}{3}}+(11-5z)^2}{\sqrt[4]{16-z}}$

## LET US SUM UP

- The statement of the principle of mathematical induction namely.

  P($n$), a statement involving a natural number $n$, is true for all $n \geq 1$, where $n$ is a fixed natural number, if

  (i) P (1) is true, and

  (ii) Whenever P($k$) is true, then P ($k$ + 1) is true for $k \in N$.

- For a natural number $n$,

  $$\left(x+y\right)^n = {}^nC_o\, x^n + {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + \ldots + {}^nC_{n-1}\, xy^{n-1} + {}^nC_n\, y^n$$

  This is called the **Binomial Theorem for a positive integral (or natural) exponent.**

- Another form of the Binomical Theorem for a positive integral exponent is $(1 + a)^n = {}^nC_o + {}^nC_1\, a + {}^nC_2\, a^2 + .... + {}^nC_{n-1}\, a^{n-1} + {}^nC_n\, a^n$

- The general term in the expansion of $(x + y)^n$ is ${}^nC_r x^{n-r} y^r$ and in the expansion of $(1 + a)^n$ is ${}^nC_r a^r$, where $n$ is a natural number and $0 \leq r \leq n$.

- If $n$ is an even natural number, there is only one middle term in the expansion of $(x + y)^n$. If $n$ is odd, there are two middle trems in the expansion.

- The formula for the general term can be used for finding the middle term(s) and some other specific terms in an expansion.

- The statement

  $$(1 + x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + ....$$

  where, $r$ is a rational number and $|x| < 1$ is called the **Binomial Theorm for a rational exponent.** In this expansion, the number of terms is infinite if $r$ is not a whole number.

- $$(x+y)^r = x^r + r\,x^{r-1}y + r\,\tfrac{(r-1)}{2!}\,x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + ....,$$

where $r$ is a rational number and $\left|\frac{y}{x}\right| < 1$, is another form of the Binomial Theorem for a rational exponent.

- Expressions like $(ax + by)^r$, where r is a rational number, can be expanded in two different ways, depending on whether $\left|\frac{by}{ax}\right| < 1$ or $\left|\frac{ax}{by}\right| < 1$.

## SUPPORTIVE WEB SITES

*http://www.wikipedia.org*

*http://mathworld.wolfram.com*

## TERMINAL EXERCISE

1. Verify each of the following statements, using the principle of mathematical induction :

(a) The number of subsets of a set with $n$ elements is $2^n$.

(b) $(a + b)^n > a^n + b^n \;\forall n \geq 2$, where $a$ and $b$ are positive real numbers.

(c) $a + ar + ar^2 + .... + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$, where r > 1 and $a$ is a real number.

(d) $(x^{2n} - 1)$ is divisible by $(x + 1)\,\forall x \in N$ .

(e) $(10^{2n-1} + 1)$ is a multiple of 11.

[**Hint:** $10^{2k+1} + 1 = 10^2(10^{2k-1} + 1) - 99$]

(f) $(4.10^{2n} + 9.10^{2n-1} + 5)$ is a multiple of 99.

(g) $n(n^2 - 1)$ is a multiple of 24, when $n$ is odd.

[Since $n$ is odd, assume that P($2k + 1$) is true, as ($2k + 3$) is always odd. Then try to prove that P($2k + 3$) is true.]

(h) $(1 + x)^n > 1 + nx$, where $x > 0$.

(i) If $f$ and $g$ are polynomials in $x$ with real coefficients and $f + g \neq 0$, then $(f + g)$ divides $(f^{\,2n-1} + g^{\,2n-1})\,\forall n \in N$.

2. Write the expansion of each of the following :

(a) $(3x + 2y)^5$ (b) $(p - q)^8$ (c) $(1 - x)^8$

(d) $\left(1+\frac{2}{3}x\right)^6$ (e) $\left(x+\frac{1}{2x}\right)^6$ (f) $\left(3x-y^2\right)^5$

(g) $\left(\frac{x^2}{4}+\frac{2}{x}\right)^4$ (h) $\left(x^2-\frac{1}{x^3}\right)^7$ (i) $\left(x^3+\frac{1}{x^2}\right)^5$

(j) $\left(\frac{1}{x^2}-x^3\right)^4$

3. Write the (r + 1)th term in the expansion of each of the following, where $n \in N$ :

(a) $(3x - y^2)^n$ (b) $\left(x^3+\frac{1}{x}\right)^n$

4. Find the specified terms in the expansion of each of the following :

(a) $(1-2x)^7$ : 3rd term [Hint : Here $r = 2$] (b) $\left(x+\frac{1}{2x}\right)^6$ : middle term (s)

(c) $(3x-4y)^6$ : 4th term (d) $\left(y^2-\frac{1}{y}\right)^{11}$ : middle term (s)

(e) $(x^3-y^3)^{12}$ : 4th term (f) $(1-3x^2)^{10}$ : middle term (s)

(g) $(-3x-4y)^6$ : 5th term (h) Write the rth term in the expansion of $(x-2y)^6$.

(i) Write the $(r-1)$ th term in the expansion of $(1+2x)^8$.

5. If $T_r$, denotes the rth term in the expansion of $(1+x)^n$ in ascending powers of $x$ ($n$ being a natural number) , prove that

$r(r+1)T_{r+2} = (n-r+1)(n-r)x^2\,T_r$

[**Hint :** $T_r = {}^nC_{r-1}\,x^{r-1}$ and $T_{r+2} = {}^nC_{r+1}\,x^{r+1}$]

6. $k_r$ is the coefficient of $x^{r-1}$ in the expansion of $(1+2x)^{10}$ in ascending powers of $x$ and $k_{r+2} = 4k_r$. Find the value of $r$.

[**Hint :** $k_r = {}^{10}C_{r-1}\,2^{r-1}$ and $k_{r+2} = {}^{10}C_{r+1}2^{r+1}$]

7. The coefficients of the 5th, 6th and 7th terms in the expansion of $(1+a)^n$ ($n$ being a natural number) are in A.P. Find $n$.

[**Hint :** ${}^nC_5 - {}^nC_4 = {}^nC_6 - {}^nC_5$]

8. Expand $(1+y+y^2)^4$. [**Hint :** $(1+y+y^2)^4 = \{(1+y)+y^2\}^4$]

9. Write the expansion of each of the following :

(a) $(1-x)^{-4}, |x|<1$ (b) $\dfrac{1}{(1+x)^3}, |x|<1$

(c) $(3-z)^{-4}, |z|<3$ (d) $\dfrac{1}{(1+3x)^{3/2}}, |x|<\dfrac{1}{3}$

10. State the condition under which the expansion of $(x–2y)^{-3}$ will be valid in ascending powers of $y$. Also write the expansion.

11. State the condition under which the expansion of $(x–3y)^{\frac{-1}{2}}$ will be valid in ascending powers of $x$. Also write the expansion.

12. Expand the following, stating the condition of $y$ under which the expansion will be valid :

(a) $\dfrac{1}{(2+y)^4}$ (b) $(3-y)^{\frac{-2}{3}}$

13. Find the value of each of the following up to three decimal places, using the necessary number of terms in the expansion:

(a) $(0.99)^{-4}$ (b) $(1.03)^{-3}$

(c) $\sqrt[3]{26}$ [**Hint :** $(26)^{\frac{1}{3}}=(27-1)^{\frac{1}{3}}$]

(d) $\sqrt[7]{127}$ [**Hint :** $(127)^{\frac{1}{7}}=(128-1)^{\frac{1}{7}}$]

(e) $\sqrt[5]{35}$ [**Hint :** $(35)^{\frac{1}{5}}=\{32+3\}^{\frac{1}{5}}$]

(f) $\sqrt[5]{31}$ [**Hint :** $(31)^{\frac{1}{5}}=\{32-1\}^{\frac{1}{5}}$]

(g) $\sqrt[3]{1001}$ [**Hint :** $(100)^{\frac{1}{3}}=(1000+1)^{\frac{1}{3}}$]

14. Assuming $y$ to be so small that $y^2$ and higher powers of $y$ can be neglected, find the value of each of the following :

(a) $(1+5y)^{-2}(1+2y)^{\frac{-3}{2}}$ (b) $\dfrac{(1-4y)^{-3}(1-2y^2)^{\frac{1}{2}}}{(4-y)^{3/2}}$

(c) $\dfrac{\sqrt{1-3y}+(1-y)^{\frac{5}{3}}}{\sqrt{4-y}}$

15. A student puts $n = 0$ in $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + x^n$ and obtains $(1+x)^0 = 1 + 0 + 0 + \dots + x^0$, i.e. 1 = 1+1 = 2. Can you detect the error in this solution?

16. Assuming that the expansions are possible, find the coefficient of $y^3$ in $(1-4y)^2(1-2y^2)^{1/2}$.

17. Prove that $(1+x+x^2+x^3+\dots)(1-x+x^2-x^3+\dots) = 1+x^2+x^4+{}^6+\dots$

[**Hint :** LHS = $(1-x)^{-1}(1+x)^{-1} = (1-x^2)^{-1}$]

## ANSWERS

### CHECK YOUR PROGRESS 8.1

1. (b), (e) and (f) are statements; (a) is not, since we have not given the range of values of $n$, and therefore we are not in a position to decide, if it is true or not. (c) is subjective and hence not a mathematical statement. (d) is a question, not a statement.

   Note that (*f*) is universally false.

2. P(1) : 6 is a factor of $1^3 + 5.1$

   P(2) : 6 is a factor of $2^3 + 5.2$

   P($k$) : 6 is a factor of $k^3 + 5k$

   P($k$+1) : 6 is a factor of $(k + 1)^3 + 5(k + 1)$

3. (a) $P(1) : 2 \geq 2$

   $P(k) : 2^k \geq k + 1$

   $P(k + 1) : 2^{k+1} \geq k + 2$

   (b) $P(1) : 1 + x \geq 1 + x$

   $P(k) : (1 + x)^k \geq 1 + kx$

   $P(k + 1): (1 + x)^{k+1} \geq 1 + (k + 1)x$

   (c) $P(1)$: 6 is divisible by 6.

   $P(k) : k(k + 1)(k + 2)$ is divisible by 6.

   $P(k + 1): (k + 1)(k + 2)(k + 3)$ is divisible by 6

   (d) $P(1): (x - y)$ is divisible by $(x - y)$.

   $P(k): (x^k - y^k)$ is divisible by $(x - y)$

   $P(k + 1): (x^{k+1} - y^{k+1})$ is divisible by $(x - y)$

   (e) $P(1): ab = ab$

   $P(k): (ab)^k = a^k b^k$

   $P(k+1): (ab)^{k+1} = a^{k+1} . b^{k+1}$

(f) $P(1): \frac{1}{5}+\frac{1}{3}+\frac{7}{15}$ is a natural number.

$P(k)$: $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

$P(k + 1)$: $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$ is a natural number.

4. (a) $P(1)$: $\frac{1}{1\times2}=\frac{1}{2}$

$$P(2): \frac{1}{1\times2}+\frac{1}{2\times3}=\frac{2}{3}$$

$$P(k): \frac{1}{1\times2}+\frac{1}{2\times3}+...+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

$$P(k+1): \frac{1}{1\times2}+...+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

(b)

$$P(1):\ 1=1^2$$

$$P(2):\ 1+3=2^2$$

$$P(k):\ 1+3+5+...+(2k-1)=k^2$$

$$P(k+1):\ 1+3+5+...+(2k-1)+\left[2(k+1)-1\right]=(k+1)^2$$

(c)

$$P(1): 1\times2<1(2)^2$$

$$P(2): (1\times2)+(2\times3)<2(3)^2$$

$$P(k): (1\times2)+(2\times3)+...+k(k+1)<k(k+1)^2.$$

$$P(x+1): (1\times2)+(2\times3)+...+(k+1)(k+2)<(k+1)(k+2)^2$$

(d)

$$P(1):\ \frac{1}{1\times3}=\frac{1}{3}$$

$$P(2):\ \frac{1}{1\times3}+\frac{1}{3\times5}=\frac{2}{5}$$

$$P(k):\ \frac{1}{1\times3}+\frac{1}{3\times5}+...+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

$$P(k+1): \frac{1}{1\times3}+\frac{1}{3\times5}+...+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

## CHECK YOUR PROGRESS 8.3

1. (a) $8a^3+12a^2b+6ab^2+b^3$

   (b) $x^{12}-18x^{10}y+135x^8y^2-540x^6y^3+1215x^4y^4-1458x^2y^5+729y^6$

   (c) $256a^4-1280a^3b+2400a^2b^2-2000ab^3+625b^4$

   (d) $a^nx^n+na^{n-1}x^{n-1}by+\frac{n(n-1)}{2!}a^{n-2}x^{n-2}b^2y^2+...+b^ny^n$

2. (a) $1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7$

   (b) $1+\frac{7x}{y}+\frac{21x^2}{y^2}+\frac{35x^3}{y^3}+\frac{35x^4}{y^4}+\frac{21x^5}{y^5}+\frac{7x^6}{y^6}+\frac{x^7}{y^7}$

   (c) $1+10x+40x^2+80x^3+80x^4+32x^5$

3. (a) $\frac{a^5}{243}+\frac{5a^4b}{162}+\frac{5a^3b^2}{54}+\frac{5a^2b^3}{36}+\frac{5ab^4}{48}+\frac{b^5}{32}$

   (b) $2187x^7-25515x^4+127575x-\frac{354375}{x^2}+\frac{590625}{x^5}-\frac{590625}{x^8}$

   $+\frac{328125}{x^{11}}-\frac{78125}{x^{14}}$

   (c) $x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$

   (d) $\frac{x^5}{y^5}+5\frac{x^3}{y^3}+10\frac{x}{y}+10\frac{y}{x}+5\frac{y^3}{x^3}+\frac{y^5}{x^5}$

4. Rs 4.96 lakh

5. 162360

6. (i) 104060401

   (ii) 96059601

   (iii) 1.061208

   (iv) 0.941192

## CHECK YOUR PROGRESS 8.4

1. (a) ${}^{n}C_{r}\,2^{n-r}\,x^{n-r}\,y^{r}$

   (b) ${}^{n}C_{r}\,2^{n-r}\,a^{2n-2r}\,(-1)^{r}$

   (c) ${}^{n}C_{r}\,(-1)^{r}\,a^{r}$

   (d) ${}^{n}C_{r}\,3^{n-r}.x^{-2r}$

2. (a) $1792y^5$

   (b) $15120x^4$

   (c) $14784a^5b^6$

   (d) 20

   (e) $35x$

## CHECK YOUR PROGRESS 8.5

1. (a) $8064x^5y^5$

   (b) $\dfrac{1120}{81}x^{12}$

   (c) 20

   (d) $-252x^{10}$

2. (a) $35a^4b^3,\ 35a^3b^4$

   (b) $4032a^5b^4, -2016a^4b^5$

   (c) $\dfrac{-105}{4}x^4y^3, \dfrac{140}{3}x^3y^4$

   (d) $\dfrac{462}{x^4}, \dfrac{462}{x^7}$

## CHECK YOUR PROGRESS 8.6

1. (a) $1 + 3p + 6p^2 + 10p^3 + ....$

   (b) $1 + 4x + 2x^2 + ....$

   (c) $1 - 6z + 3z^2 + 4z^3 + ....$

2. (a) $\frac{1}{9}+\frac{4x}{243}+\frac{20}{6561}x^2+....$

(b) $\frac{1}{8a^3}-\frac{3x}{16a^4}+\frac{3x^2}{16a^5}-....$

(c) $3^{\frac{1}{7}}y^{\frac{1}{7}}\ [1+\frac{2}{21y}-\frac{4}{147y^2}+....]$

3. (a) (i) $\left|\frac{x}{2y}\right|<1: \frac{1}{32y^5}-\frac{5x}{64y^6}+\frac{15x^2}{128y^7}-....$

(ii) $\left|\frac{2y}{x}\right|<1: \frac{1}{x^5}-\frac{10y}{x^6}+\frac{60y^2}{x^7}-....$

(b) $|y|<\frac{1}{2}: 3^{-4/3}-8(3)^{-7/3}y+56(3)^{-10/3}y^2+\dots$

**CHECK YOUR PROGRESS 8.7**

1. (a) 1.041

(b) 0.971

(c) 1.130

(d) 1.968

(e) 3.009

(f) 0.204

2. (a) $\frac{1}{243}-\frac{10z}{729}$

(b) $1+12z$

(c) $1-\frac{5z}{6}$

(d) $61-\frac{10409}{192}z$

## TERMINAL EXERCISE

2. (a) $243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240xy^4 + 32y^5$

(b) $p^8 - 8p^7q + 28p^6q^2 - 56p^5q^3 + 70p^4q^4 - 56p^3q^5 + 28p^2q^6 - 8pq^7 + q^8$

(c) $1 - 8x + 28x^2 - 56x^3 + 70x^4 - 56x^5 + 28x^6 - 8x^7 + x^8$

(d) $1 + 4x + \dfrac{20}{3}x^2 + \dfrac{160}{27}x^3 + \dfrac{80}{27}x^4 + \dfrac{64}{81}x^5 + \dfrac{64}{729}x^6$

(e) $x^6 + 3x^4 + \dfrac{15}{4}x^2 + \dfrac{5}{2} + \dfrac{15}{16x^2} + \dfrac{3}{16x^4} + \dfrac{1}{64x^6}$

(f) $243x^5 - 405x^4y^2 + 270x^3y^4 - 90x^2y^6 + 15xy^8 - y^{10}$

(g) $\dfrac{x^8}{256} + \dfrac{x^5}{8} + \dfrac{3}{2}x^2 + \dfrac{8}{x} + \dfrac{16}{x^4}$

(h) $x^{14} - 7x^9 + 21x^4 - \dfrac{35}{x} + \dfrac{35}{x^6} - \dfrac{21}{x^{11}} + \dfrac{7}{x^{16}} - \dfrac{1}{x^{21}}$

(i) $x^{15} + 5x^{10} + 10x^5 + 10 + \dfrac{5}{x^5} + \dfrac{1}{x^{10}}$

(j) $\dfrac{1}{x^8} - \dfrac{4}{x^3} + 6x^2 - 4x^7 + x^{12}$

3. (a) $(-1)^r\ {}^nC_r 3^{n-r} x^{n-r} y^{2r}$

(b) ${}^nC_r x^{3n-4r}$

4. (a) $84x^2$

(b) $\dfrac{5}{2}$

(c) $-34560\ x^3y^3$

(d) $-462y^7,\ 462y^4$

(e) $-220x^{27}y^9$

(f) $-61236\ x^{10}$

(g) $34560x^2y^4$

(h) $(-2)^{r-1}\ {}^6C_{r-1}\ x^{7-r} y^{r-1}$

(i) $-2^{r-2}\ {}^8C_{r-2}\ x^{r-2}$

6. 5 7. 7, 14

8. $1 + 4y + 10y^2 + 16y^3 + 19y^4 + 16y^5 + 10y^6 + 4y^7 + y^8$

9. (a) $1 + 4x + 10x^2 + \ldots$

(b) $1 - 3x + 6x^2 - 10x^3 + \ldots$

(c) $\frac{1}{81} + \frac{4}{243}z + \frac{10}{729}z^2 + \ldots$

(d) $1 - \frac{9}{2}x + \frac{135}{8}x^2 - \frac{945}{16}x^3 + \ldots$

10. $\left|\frac{2y}{x}\right| < 1 : \frac{1}{x^3} + \frac{6y}{x^4} + \frac{24y^2}{x^5} + \ldots$

11. $\left|\frac{x}{3y}\right| < 1 : \frac{1}{\sqrt{-3y}} + \frac{x}{6y\sqrt{-3y}} + \frac{x^2}{24y^2\sqrt{-3y}} + \ldots$

12. (a) $\frac{1}{16} - \frac{y}{8} + \frac{5y^2}{32} - \frac{5y^3}{32} + \ldots \ |y| < 2$

(b) $\frac{1}{3^{2/3}} + \frac{2x}{9 \times 3^{2/3}} + \frac{5x^2}{81 \times 3^{2/3}} + \ldots, |y| < 3$

13. (a) 1.041

(b) 0.915

(c) 2.833

(d) 1.998

(e) 2.037

(f) 1.987

(g) 10.003

14. (a) 1–13y

(b) $\frac{8+99y}{64}$

(c) $1 - \frac{35}{24}y$

15. Expansion is valid when $n$ is a natural number. Here, $n = 0$

16. 8.