Hooke s law and its consequences[1]

Historically, the notion of elasticity was first announced in 1676 by Robert Hooke (1635–1703) in the form of an anagram, *ceiinosssttuv*. He explained it in 1678 as *Ut tensio sic vis*, or "the power of any springy body is in the same proportion with the extension." Hooke's law is the constitutive law for a *Hookean*, or *linear elastic*, material. As stated in the original form, its meaning is not very clear. One way to give it precise meaning is to make use of the common notion of "springs," and consider the load–displacement relationship. The second way is to state it in as an equation relating strain and stress, as is done in the mathematical theory of elasticity. The first way of "springs" is followed here.

Consider the static equilibrium state of a solid body under the action of external forces, as is shown in Fig. 1. Let the body be supported in a manner that rigid body motion is impossible, and let it be described with respect to a rectangular Cartesian reference frame. Consider the three hypotheses

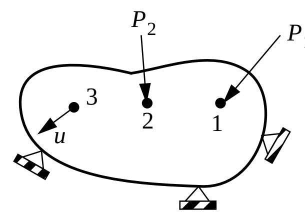


Fig. 1. Static equilibrium of a body under external forces

1. The body is a continuum.

Under this hypothesis the atomic structure of the body is ignored and the body is idealized into a geometrical copy in Euclidean space whose points are identified with the material particles of the body. The continuity is defined in the mathematical sense with respect to the idealized continuum. Neighboring points remain neighbors under any loading condition. No cracks or holes may open up in the interior of the body under the action of external load.

To introduce the second hypothesis, consider the action of a set of forces on the body. Let the forces be fixed in direction and in point of application, and let the magnitudes of all the forces be increased or decreased together: always bearing the same ratio to each other. Let the forces be denoted by $\vec{P}_1, \vec{P}_2, \ldots, \vec{P}_n$ and their magnitude by $P_1, P_2, \ldots, P_n$. Then, the ratios $P_1{:}P_2{:}P_3{:}\ldots{:}P_n$ remain fixed. When such a system of forces is applied on the body, the body deforms. Let the displacement at an arbitrary point in an arbitrary direction, denoted by u, be measure with respect to a rectangular Cartesian reference frame. The second hypothesis is

2. Hooke's law

$$u = a_1 P_1 + a_2 P_2 + \ldots + a_n P_n \qquad \textbf{(1)}$$

where $a_1, a_2, \ldots, a_n$ are constants independent of the magnitude of $P_1, P_2, \ldots, P_n$. The constants $a_1, a_2, \ldots, a_n$ depend on the point where the displacement is measured and on the directions and points of application of the forces. Hooke's law in the form of Eq. (1) can be subjected to experimental examination.

To complete the formulation of elasticity, we need a third hypothesis:

3. There exists a unique unstressed state of the body, to which the body returns whenever all external forces are removed.

A number of deductions can be drawn from these hypotheses.

1. Much of this discussion is adapted from: Y.C. Fung, *Foundations of Solid Mechanics*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey,1965, pp. 1-7.

A. The principle of superposition for loads at the same point.

Let $\vec{P}_1$ and $\vec{P}_1'$ be forces in the same direction acting through the same point. Then the resultant displacement u is equal to the sum of the displacements produced by $\vec{P}_1$ and $\vec{P}_1'$ acting individually, regardless of the order of application of $\vec{P}_1$ and $\vec{P}_1'$. This is evident from hypothesis 2 when applied to one load acting on the body, since constant a_1 is independent of $\vec{P}_1$ and $\vec{P}_1'$.

B. Principle of superposition

By a combination of hypotheses 2 and 3, we can show that Eq. (1) is valid not only for a system of loads for which the ratios $P_1:P_2:\ldots:P_n$ remain fixed as originally assumed, but also for an arbitrary set of loads $\overrightarrow{P_1}, \overrightarrow{P_2}, \ldots, \overrightarrow{P_n}$. In other words, Eq.(1) holds regardless of the order in which the loads are applied. The constant a_1 is independent of the loads $\overrightarrow{P_2}, \overrightarrow{P_3}, \ldots, \overrightarrow{P_n}$. The constant a_2 is independent of the loads $\overrightarrow{P_1}, \overrightarrow{P_3}, \ldots, \overrightarrow{P_n}$; etc. This the principle of superposition of the load-and-deflection relationship.

- *Proof*: If a proof of the statement above can be established for an arbitrary pair of loads, then the general theorem can be proved by mathematical induction.

  Let $\vec{P}_1$ and $\vec{P}_2$, with magnitudes P_1 and P_2, be a pair of arbitrary loads acting at points 1 and 2, respectively. Let the displacement in a specific direction be measured at point 3. See Fig. 1. According to hypothesis 2, if $\vec{P}_1$ is applied alone, then at point 3 a displacement $u_3 = c_{31}P_1$ is produced. If $\vec{P}_2$ is applied alone, a displacement $u_3 = c_{32}P_2$ is produced. If $\vec{P}_1$ and $\vec{P}_2$ are applied together, with ratio P_1/P_2 fixed, then according to hypothesis 2 the displacement can be written as

  $$u_3 = c'_{31}P_1 + c'_{32}P_2 \qquad \textbf{(a)}$$

  The question arises whether $c'_{31} = c_{31}$ and $c'_{32} = c_{32}$. The answer is affirmative, as can be shown as follows. After $\vec{P}_1$ and $\vec{P}_2$ are applied, we take away $\vec{P}_1$, This produces a change in displacement $-c''_{31}P_1$, and the total displacement is

  $$u_3 = c'_{31}P_1 + c'_{32}P_2 - c''_{31}P_1 \qquad \textbf{(b)}$$

  Now only $\vec{P}_2$ acts on the body. Hence, upon unloading $\vec{P}_2$, we shall have

  $$u_3 = c'_{31}P_1 + c'_{32}P_2 - c''_{31}P_1 - c_{32}P_2 \qquad \textbf{(c)}$$

  Now all the loads are removed, and u_3 must vanish according to hypothesis 3. Rearranging terms, we have

  $$(c'_{31} - c''_{31})P_1 = (c_{32} - c'_{32})P_2 \qquad \textbf{(d)}$$

  Since the only possible difference of c'_{31} and c''_{31} must be caused by the action of $\vec{P}_2$, the difference $c'_{31} - c''_{31}$ can only be a function of $\vec{P}_2$ and not of $\vec{P}_1$. Similarly, $c_{32} - c'_{32}$ can only be a function of $\vec{P}_1$. If we write Eq. (d) as

  $$\frac{c'_{31} - c''_{32}}{P_2} = \frac{c_{32} - c'_{32}}{P_1} \qquad \textbf{(e)}$$

then the left-hand side is a function of P_2 alone, and the right-hand side is a function of P_1 alone. Since P_1 and P_2 are arbitrary numbers, the only possibility for Eq. (e) to be valid is for both sides equal to a constant k, which is independent of both P_1 and P_2. Hence,

$$c'_{32} = c_{32} - kP_1 \tag{f}$$

But a substitution of Eq. (f) into (a) yields

$$u_3 = c'_{31}P_1 + c_{32}P_2 - kP_1P_2 \tag{g}$$

The last term is nonlinear in P_1 and P_2, and Eq. (g) will contradict hypothesis 2 unless k vanishes. Hence, $k = 0$ and $c'_{32} = c_{32}$. An analogous procedure shows $c'_{31} = c''_{31} = c_{31}$.

Thus, the principle of superposition is established for one and two forces. An entirely similar procedure will show that if it valid for *m* forces, it is also valid for *m+1* forces. Thus, the general theorem follow by mathematical induction. ■

The constants c_{31}, c_{32}, etc., are seen to be of significance in defining the elastic property of the solid body. They are called *flexibility influence coefficients*.

C. Corresponding forces and displacements and the unique meaning of the total work

Let us now consider a set of external forces $\vec{P}_1, \vec{P}_3, \ldots, \vec{P}_n$ acting on the body and *define the set of displacements at the points of application and in the direction of the loads as the displacements "corresponding" to the forces at these points*. The reactions at the points of support are considered as external forces exerted on the body and included in the set of forces.

Under the loads $\vec{P}_1, \vec{P}_3, \ldots, \vec{P}_n$, the corresponding displacements may be written as

$$\begin{aligned} u_1 &= c_{11}P_1 + c_{12}P_2 + \ldots + c_{1n}P_n \\ u_2 &= c_{21}P_1 + c_{22}P_2 + \ldots + c_{2n}P_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_n &= c_{n1}P_1 + c_{n2}P_2 + \ldots + c_{nn}P_n \end{aligned} \tag{2}$$

If we multiply the first equation by P_1, the second equation by P_2, etc., and add, we obtain

$$\begin{aligned} P_1u_1 + P_2u_2 + \ldots + P_nu_n = {} & c_{11}P_1^2 + c_{12}P_1P_2 + \ldots + c_{1n}P_1P_n + \\ & c_{21}P_1P_2 + c_{22}P_2^2 + \ldots + c_{2n}P_2P_n + \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ & c_{n1}P_1P_n + c_{n2}P_2P_n + \ldots + c_{nn}P_n^2 \end{aligned} \tag{3}$$

The quantity above is independent of the order in which the loads are applied. Hence, it has a definite meaning for each set of loads $\vec{P}_1, \vec{P}_3, \ldots, \vec{P}_n$.

Now, in a special case, the meaning of the quantity on the left-hand side of Eq. (3) is clear. This is the case in which the ratios $P_1 : P_2 : P_3 : \ldots : P_n$ are kept constant and the loading increases very slowly from zero to the final value. In this case, the corresponding displacements also increase proportionally and slowly. It should be clear that the work done by the force $\vec{P}_1$ is exactly $\frac{1}{2} P_1 u_1$, that of $\vec{P}_2$ is $\frac{1}{2} P_2 u_2$, etc. See Fig. 2.

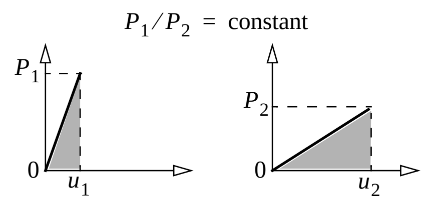


Fig. 2. Load-displacement plots for proportional loading

Hence, we conclude from Eq. (3) that the *total work done, W, by the set of forces is independent of the order in which the forces are applied.*

$$W = \frac{1}{2} \sum_{i=1}^{n} P_i u_i \tag{4}$$

D. Maxwell's reciprocal theorem

An important property of the influence coefficients of corresponding displacements follows immediately.

The influence coefficients for corresponding forces and displacements are symmetric.

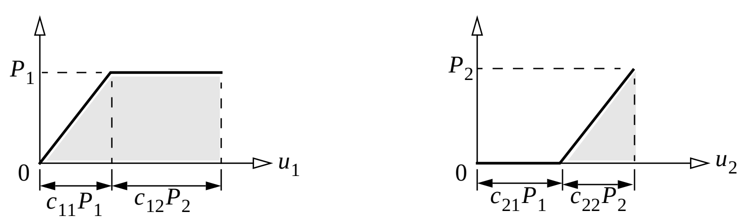
$$c_{ij} = c_{ji} \tag{5}$$

In other words, *the displacement at point i due to a unit load at another point j is equal to the displacement at j due to a unit load at i, provided that the displacements and forces "correspond," i.e., that they are measured in the same direction at each point.*

- *Proof*: Consider two forces $\vec{P}_1$ and $\vec{P}_2$. First, apply $\vec{P}_1$ slowly with $\vec{P}_2 = 0$. At the final value of $\vec{P}_1$, the displacement of point 1 is $c_{11} P_1$ and the displacement of point 2 is $c_{21} P_1$. The work done is $\frac{1}{2} c_{11} P_1^2$. With $\vec{P}_1$ held fixed, apply $\vec{P}_2$ slowly until $\vec{P}_2$ attains its final value. The additional displacement at point 1 is $c_{12} P_2$ and the additional displacement at point 2 is $c_{22} P_2$. The additional work done is $P_1 c_{12} P_2 + \frac{1}{2} c_{22} P_2^2$. When the forces are applied in the order $\vec{P}_1, \vec{P}_2$ the total work done, as shown in Fig. 3, is

$$W = \underbrace{\frac{1}{2} c_{11} P_1^2 + c_{12} P_1 P_2}_{\text{pt. 1}} + \underbrace{\frac{1}{2} c_{22} P_2^2}_{\text{pt. 2}}$$

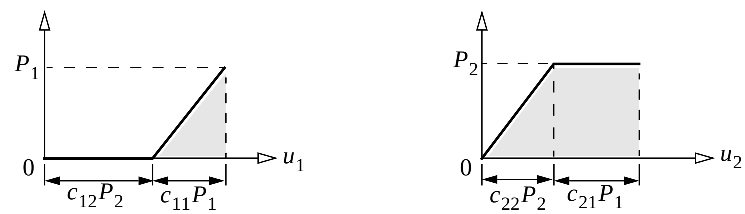
Fig. 3. Load-displacement plots for the loading sequence P_1, P_2.



Second, apply $\vec{P}_2$ slowly with $\vec{P}_1 = 0$. At the final value of $\vec{P}_2$, the displacement of point 1 is $c_{12}P_2$ and the displacement of point 2 is $c_{22}P_2$. The work done is $\frac{1}{2}c_{22}P_2^2$. With $\vec{P}_2$ held fixed, apply $\vec{P}_1$ slowly until $\vec{P}_1$ attains its final value. The additional displacement at point 1 is $c_{11}P_1$ and the additional displacement at point 2 is $c_{21}P_1$. The additional work done is $P_2c_{21}P_1 + \frac{1}{2}c_{11}P_1^2$. When the forces are applied in the order $\vec{P}_2, \vec{P}_1$ the total work done, as shown in Fig. 4, is

$$W' = \underbrace{\frac{1}{2}c_{11}P_1^2}_{\text{pt. 1}} + \underbrace{\frac{1}{2}c_{22}P_2^2 + c_{21}P_1P_2}_{\text{pt. 2}}$$

Fig. 4. Load-displacement plots for the loading sequence P$_2$, P$_1$.



But according to deduction (C) above, $W = W'$ for arbitrary P_1, P_2. Hence, $c_{12} = c_{21}$, and the theorem is proved. ■