The Fundamental Theorem of Algebra

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Sometimes a theorem is of such importance that it becomes known as the fundamental theorem in a branch of mathematics. This paper will explore the mathematical ideas that one needs to understand to appreciate the Fundamental Theorem of Algebra. The Fundamental Theorem of Algebra (FTA) encompasses one of the big ideas in mathematical discourse and has led to its own branch of mathematics, mainly the branch involving roots of polynomials. One of the big questions we ask ourselves when dealing with polynomials is "how many zeros (roots) does the function have". The FTA tells us that every complex polynomial of degree n has precisely n complex roots, although there are many variations to this statement that basically mean the same general idea. It can be very helpful in mathematics to find roots of polynomials and the FTA is centered on this idea. In order to understand the FTA fully and know exactly what is meant, some background information needs to be reviewed and explained in detail so one may understand how the FTA was developed and is used today.

Polynomials

Since the FTA deals with polynomials, it makes sense that we should have an excellent understanding of polynomial functions and their properties. Polynomials are used in a wide range of problems, from elementary word problems to calculus, chemistry, economics, physics, and numerical analysis. In order to understand the make-up of polynomials, one first needs to understand what classifies as a monomial.

Monomial Definition. A single term that consists of a number, a variable, or the product of a number and variable to a positive integer exponent is a monomial.

Examples: 5, -24, x, 4x, and $3x^3$

Understanding monomials helps one to understand polynomials because a monomial is a polynomial with only one term. We define a polynomial as follows:

Polynomial Definition. A polynomial is the addition or subtraction of monomials written in the form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_2 x^2 + a_1 x + a_0$$

where n is a non-negative integer and $a_0, a_1, a_2, ..., a_n$ are constant coefficients.

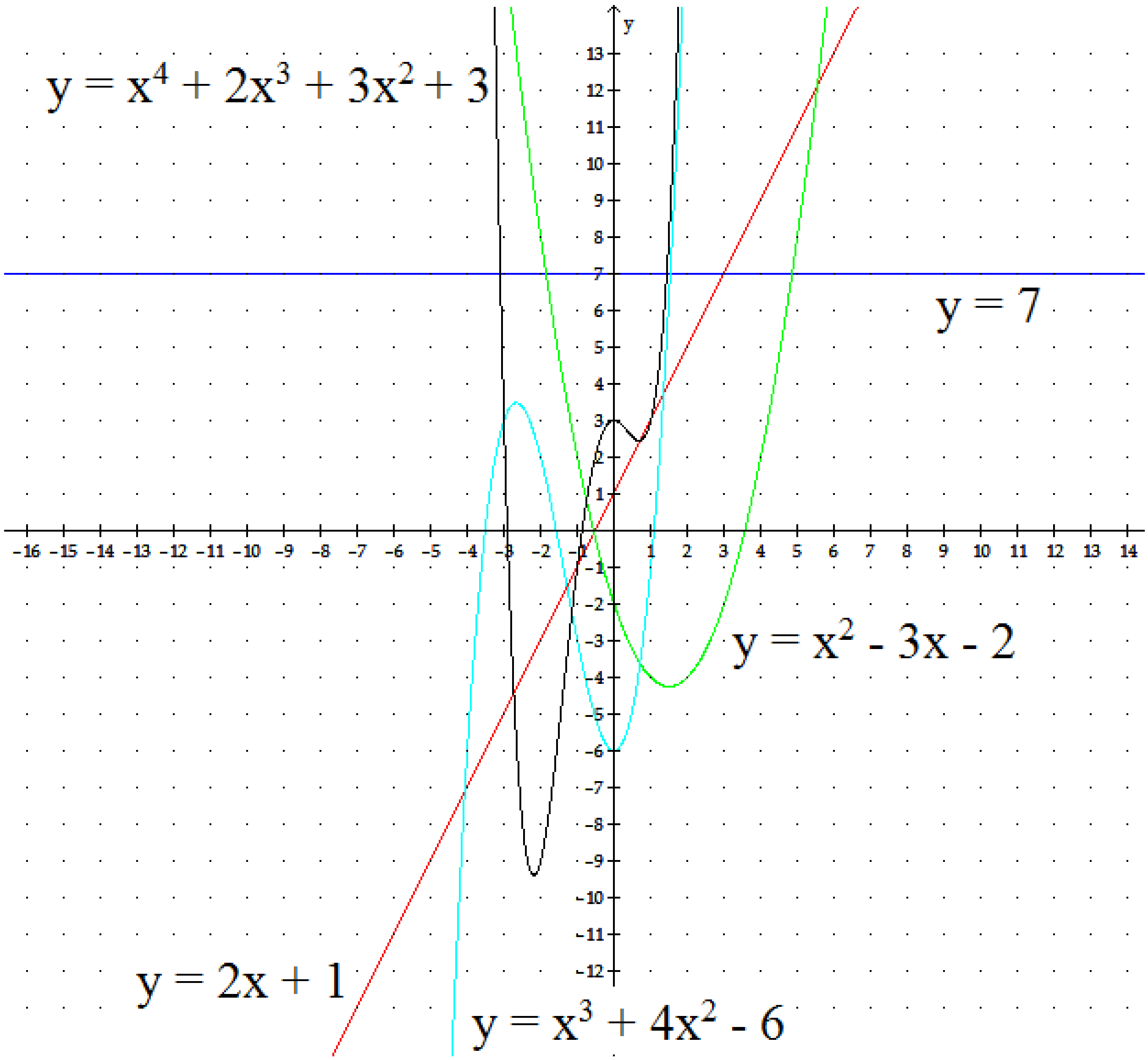
Examples: $g(x) = 3x^4 + 2x^3 - 6x^2 + 7x - 3$ and $h(x) = x^4 + 2x - 1$

Polynomials can also be given specific names to identify the number of terms such as monomial (1 term), binomial (2 terms), and trinomial (3 terms). Polynomials with more than 3 terms are usually just referred to as polynomials. Examples of binomials are $f(x) = 4x + 5$ and $g(x) = 5x^2 + 2$. Notice that each individual term satisfies the definition of a monomial and then is joined to another monomial by either addition or subtraction.

The next idea associated with polynomials that one needs to understand in order to comprehend the FTA is the degree of a polynomial. The degree of a polynomial is found by identifying the largest exponent. Therefore, if you had a binomial like $f(x) = 5x^4 + 3x$, its degree is 4 because the largest exponent is 4. In class, we have witnessed and used polynomials of different degrees. A polynomial such as $f(x) = 3x + 4$ is linear and has a degree of 1. Another polynomial such as $g(x) = x^2 + 4x - 5$ is a quadratic function and has degree 2. Usually terms of the polynomial are written in descending order of degree, although they can be written in any order. Some polynomials have special names based on their degrees and they are:

Degree	*Name*	*Example*
0	Constant	$y = 7$
1	Linear	$y = 2x + 1$
2	Quadratic	$y = x^2 - 3x - 2$
3	Cubic	$y = x^3 + 4x^2 - 6$
4	Quartic	$y = x^4 + 2x^3 + 3x^2 + 3$
5	Quintic	$y = 25x^5$

The graph below shows how some of the examples would graph on a coordinate plane.



We will now take a closer look at how some of the basic mathematical operations (addition, subtraction, multiplication, and division) work on polynomials. The addition or subtraction of two polynomials is a commonly used mathematical operation in algebra classes today. The rule for adding or subtracting polynomials is that you add or subtract the coefficients of all of the terms that share a variable of the same power and the new polynomial's degree will be the same or smaller than the largest degree polynomial term in the sum. An example is $(x^2+2x-5)+(3x^3-4x+8)$ and the resulting polynomial is $3x^3+x^2-2x+3$ and thus the degree of it is the same as the largest degree polynomial term in the sum. Another example is $(2x^2+6x+5)+(-2x^2-5x+3)$ and the resulting polynomial is $x+8$ and thus the degree of this polynomial is smaller than the largest degree polynomial term. There will never be a case where the new degree will be larger than any of the degrees of the polynomial terms when adding or subtracting polynomials.

The product of two polynomials is another common operation in algebra classes and so we know this holds true, but what are the rules? When multiplying two polynomials, the following explains why degree of the new polynomial is the sum of the degrees of the two polynomials.

Product of Polynomials: If two polynomials are multiplied together, the degree of the new polynomial will be the sum of the degrees of the two polynomials written as:

$$p(x) = d(x) \cdot f(x)$$

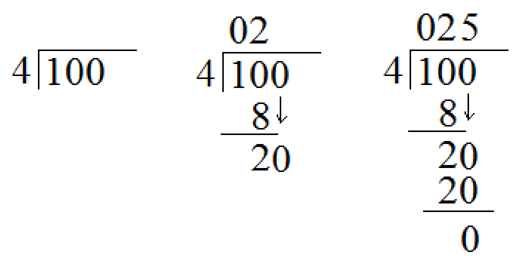
Degree of *p(x)* = (Degree of *d(x)*) + (Degree of *f(x)*)

An example would be if you had $(2x^2+5)\cdot(5x+3)$ and multiplied by using the distributive property to get $10x^3+6x^2+25x+15$. Since the first polynomial had a degree of 2 and the second polynomial had a degree of 1, they combine to make a polynomial of degree 3. This concept holds true for the multiplication of all polynomials because unlike the operations of addition and subtraction, with multiplication there is no way to cancel the highest degree terms.

So is there division of polynomials? Division is the inverse operation to multiplication. The question is, when you divide two polynomials, do you get another polynomial? The answer is yes, in some cases, but it does not always hold true. At this point we need to shift away from polynomials for a quick review of other key concepts before explaining how to divide polynomials and why the division of polynomials does not always produce another polynomial.

Other Operations

In order build up to the FTA, it is essential that we understand some of the other mathematical terms and operations. To begin with, we will review the division algorithm for integers. Most people will remember the division algorithm as something learned in grade school; given an example of 100 divided by 4, you would have seen something like the picture to the right. This algorithm works for all numbers, but the question is why exactly does this model work and what happens when the numbers divided do not come out evenly? This even works when things do not divide evenly because you end up with remainders or decimal answers. The division algorithm as presented in the elementary model above is derived from the following theorem:



Division Algorithm Given two integers a and d, with $d \neq 0$, there exist unique integers q and r such that $a = qd + r$ and $0 \leq r < |d|$, where $|d|$ denotes the absolute value of d. In this case:

- a is the dividend
- q is the quotient
- d is the divisor
- r is the remainder

Since r is a remainder, it makes sense that r must be between 0 and the absolute value of the divisor because if r was bigger than that, then the divisor would go into the number one more time. Therefore, if you take the dividend and divide by the divisor, you get the quotient and a possible remainder left over. An example is 20 divided by 17 which looks like this: $17\overline{)20}$

$a = qd + r \qquad 20 = q17 + r \qquad 20 = (1)17 + 3$

Thus, we would get a quotient of 1 with a remainder of 3, and if we wanted to find the decimal approximation, we would continue the same type of steps, but some manipulation would need to be done, as the following illustrates.

We begin with our answer of $1+\frac{3}{17}$ and rewrite this as $1+\frac{1}{10}\left(\frac{30}{17}\right)$. By rewriting, we can now find what $\frac{30}{17}$ is by the same method as before. Thus, we get $30=(1)17+13$ and get 1 with a remainder of 13. Therefore, our new answer is $1+\frac{1}{10}\left(1+\frac{13}{17}\right)$ because we replaced $\frac{30}{17}$ with $1+\frac{13}{17}$. Next we can distribute and get $1+\frac{1}{10}+\frac{1}{10}\left(\frac{13}{17}\right)$ and now we need to rewrite so we can divide $\frac{13}{17}$, and so, we get $1+\frac{1}{10}+\frac{1}{10}\left(\frac{1}{10}\cdot\frac{130}{17}\right)=1+\frac{1}{10}+\frac{1}{100}\left(\frac{130}{17}\right)$. We can now use the division algorithm for $\frac{130}{17}$ and get $130=(7)17+11$ and this gives us 7 with a remainder of 11. After making this substitution, we have $1+\frac{1}{10}+\frac{1}{100}\left(7+\frac{11}{17}\right)=1+\frac{1}{10}+\frac{7}{100}+\left(\frac{1}{100}\cdot\frac{11}{17}\right)$ and this of course is $1.17+\frac{11}{1700}$. We could continue the division algorithm and eventually take this to the number of digits we prefer or until the remainder is zero. The long division algorithm is a shortened version of this. The process is very important when dealing with polynomials as we shall see soon.

The next important topic to discuss when trying to fully understand the FTA is the number system and the definition of a "field". The commonly used number systems are defined as follows:

N = Natural Numbers	**N = {1, 2, 3, 4, 5, 6, …}** (sometimes called the counting numbers)
W = Whole Numbers	**W = {0, 1, 2, 3, 4, 5, …}** (i.e. we include zero)
Z = Integers	**Z = {… -4, -3, -2, -1, 0, 1, 2, 3, 4, …}** (i.e. we include negatives)
Q = Rational Numbers	**Q = {** $\frac{a}{b}$ **| *a, b* are in Z and** $b \neq 0$ **}** (i.e. fractions)
I = Irrational Numbers	**I** = has non-repeating and non-terminating decimals (i.e $\pi, \sqrt{2}, e$)
R = Real Numbers	The rational numbers together with the irrational numbers.

Wikipedia notes that the concept of a field was used implicitly by Niels Henrik Abel and Evariste Galois in their work on the solvability of polynomial equations with rational coefficients of degree 5 or higher. A field is an algebraic structure with notions of addition and multiplication, which satisfy certain axioms. The axioms are things like closure, commutative, and associative properties and are listed in more depth and defined below. Also indicated below are the number systems that satisfy the axioms. Here are the common "field axioms" that every field X satisfies.

	Where true
<u>Addition Properties</u>	
• **X** is **closed** with respect to addition.	**N, W, Z, Q, R**
o If a, b $\in$ X, then a + b $\in$ X	
• Addition in **X** is **commutative**.	**N, W, Z, Q, R**
o If a, b $\in$ X, then a + b = b + a	
• Addition in **X** is **associative**.	**N, W, Z, Q, R**
o If a, b, c $\in$ X, then (a + b) + c = a + (b + c)	
• **X** has an **additive identity**, namely 0.	**W, Z, Q, R**
o If a $\in$ X, then a + 0 = 0 + a = a	
• *In **X** every element has an **additive inverse**.*	**Z, Q, R**
o If a $\in$ X, then there exists –a $\in$ X so that a + –a = –a + a = 0	
<u>Multiplication Properties</u>	
• **X** is **closed** with respect to multiplication.	**N, W, Z, Q, R**
o If a, b $\in$ X, then a • b $\in$ X	

- Multiplication in **X** is **commutative**. **N, W, Z, Q, R**
  - If a, b $\in$ X, then a $\bullet$ b = b $\bullet$ a.
- Multiplication in **X** is **associative**. **N, W, Z, Q, R**
  - If a, b, c $\in$ X, then (a $\bullet$ b) $\bullet$ c = a $\bullet$ (b $\bullet$ c)
- **X** has a **multiplicative identity**, namely 1. **N, W, Z, Q, R**
  - If a $\in$ X, then a $\bullet$ 1 = 1 $\bullet$ a = a
- *In **X** every element (except zero) has a **multiplicative inverse.*** **Q, R**
  - If $a \in$ X, $a \neq 0$, then there exists $1/a \in$ X so that a $\bullet$ $1/a$ = $1/a$ $\bullet$ a = 1

<u>Addition and multiplication are connected</u>

- **In X, multiplication distributes over addition**. **N, W, Z, Q, R**
  - If a, b, c $\in$ X, then a$\bullet$(b + c) = a$\bullet$b + a$\bullet$c

If all 11 properties hold true, then we say the set of numbers along with the given operations form a field. From above we can tell that the rational and real numbers form a "field". In certain instances, like when solving equations, some fields are just not big enough. To solve the equation $3x+4=5$, we can stay in the field of rational numbers because we get an answer of x=1/3. But if we have an equation like $x^2 = 2$, we get an answer of x=$\pm\sqrt{2}$. This is not rational and thus the equation has a solution in the field of real numbers. Not even the real field is large enough sometimes because the equation $x^2 + 3 = 2$ doesn't have a real solution and thus we introduce the field of complex or imaginary numbers in which it does have a solution.

So how does this idea of a field apply to our investigation of the FTA? One thing to understand is that addition, subtraction, and multiplication hold true for the integers, even though the integers are not a field. The same three properties hold true for polynomials as well, as was mentioned earlier when we defined polynomials. But what about division and why are neither integers nor polynomials closed under division? The property of multiplicative inverses doesn't hold true for integers because you can take an integer and divide by another integer and get a rational number that is not an integer (for example $\frac{2}{3}$). Therefore integers are not closed under division. To understand why divisibility does not always work for polynomials, we will take a look at a few examples.

One way to divide polynomials is through the use of long division. This method allows us to take a polynomial (dividend) and divide by another polynomial (divisor). Just like the division algorithm presented earlier of $a = qd + r$ was used as the model to divide numbers, we can modify this slightly to divide polynomials.

Division Algorithm for Polynomials

Given two polynomials *p(x)* and *d(x),* where *d(x)* is non-zero, then there exists polynomials q(x) and r(x) such that:

$$p(x) = q(x) \cdot d(x) + r(x)$$

- *p(x)* is the dividend
- *q(x)* is the quotient
- *d(x)* is the divisor
- *r(x)* is the remainder where $0 \leq$ degree of *r(x)* $<$ degree of *d(x)*

For example if we wanted to let $p(x) = 3x^4 - 2x + 5$ and divide by $d(x) = x^2 + x - 7$

$$x^2 + x - 7 \overline{\big) 3x^4 - 2x + 5}$$

$$\begin{array}{r|l} & 3x^2 \\ \hline x^2 + x - 7 & 3x^4 - 2x + 5 \\ & 3x^4 + 3x^3 - 21x^2 \\ \hline & -3x^3 + 21x^2 - 2x + 5 \end{array}$$

This means $3x^4 - 2x + 5 = 3x^2\left(x^2 + x - 7\right) + \left(-3x^3 + 21x^2 - 2x + 5\right)$ and we can now do the same thing to try and cancel the next term.

$$\begin{array}{r} 3x^2-3x \qquad\qquad \\ x^2+x-7\overline{)\,3x^4-2x+5 \qquad} \\ 3x^4+3x^3-21x^2 \qquad\qquad \\ \hline -3x^3+21x^2-2x+5 \\ -3x^3-3x^2+21x \qquad \\ \hline 24x^2-23x+5 \end{array}$$

This means $3x^4-2x+5=(3x^2-3x)(x^2+x-7)+(24x^2-23x+5)$ and we can now do the same thing to try and cancel the next term.

$$\begin{array}{r} 3x^2-3x+24 \qquad\qquad \\ x^2+x-7\overline{)\,3x^4-2x+5 \qquad} \\ 3x^4+3x^3-21x^2 \qquad\qquad \\ \hline -3x^3+21x^2-2x+5 \\ -3x^3-3x^2+21x \qquad \\ \hline 24x^2-23x+5 \\ 24x^2+24x-168 \\ \hline -47x+173 \end{array}$$

This means $3x^4-2x+5=(3x^2-3x+24)(x^2+x-7)+(-47x+173)$, where the remainder is $-47x+173$. There are certain instances where dividing polynomials can also give us a remainder of zero, as in the case of $p(x)\div d(x)$ where $p(x)=x^2-3x+2$ and $d(x)=x-2$. This division gives us an answer of $x-1$ and we have no remainder. What this really tells us is that $(x-2)(x-1)=x^2-3x+2$, so that $(x-2)$ is a factor of x^2-3x+2. From these examples, we see that it would be helpful to know what polynomials divide a polynomial evenly. The method of how to find these factors may be more apparent in the last example because it is written in a form we have talked about in class. If we set the equation x^2-3x+2 equal to zero and solve for the solutions that make this true, we are doing something called finding the zeros. In this case we get $x^2-3x+2=(x-2)(x-1)=0$ and the roots or solutions or zeros are x = 2 and x = 1. Therefore, finding the roots of polynomials will essentially help us to identify factors of them that we would like to use as our divisor.

This brings us to the concept of a polynomial being irreducible. Irreducible can be best described by the following definition.

Definition of Irreducible Polynomial: A polynomial is irreducible over a field K if it cannot be written as the product of two polynomials of lesser degree whose coefficients come from K.

This basically means that a polynomial is irreducible if it has been factored completely. For instance, $x^2 + x$ can be factored into $x(x+1)$ and thus this polynomial is not irreducible. The hardest part about factoring polynomials and determining if they are irreducible is that this is not often obvious. The polynomial $x^2 - 28$ looks irreducible (and is, in fact, irreducible over the rational numbers), however, over the real numbers it can be factored as $\left(x+\sqrt{28}\right)\cdot\left(x-\sqrt{28}\right)$.

The polynomial $x^2 + 1$ again is irreducible over the rational numbers (and so, "looks" irreducible) but if we allow this to be viewed over the complex field, then it can be factored into $\left(x+\sqrt{-1}\right)\cdot\left(x-\sqrt{-1}\right)$. Some examples of irreducible polynomials over the real field include $x+1$ and $x^2 + x + 1$. We will discuss irreducible polynomials in more depth after discussing how roots will help us.

So what if we have $f(x) = 3x^3 - 4x^2 + 5x + 6$, and we want to find something that divides it evenly so we do not get a remainder. Is there even a polynomial that exists that is a factor of $f(x)$? We do know that since the degree of a product of polynomials is the sum of the degrees of the factors, and since the degree of f(x) is three, any factorization into polynomials of lower degree must have one factor be linear. This basically means if you were to factor a polynomial with leading coefficient of x^3, it must factor into x and x^2 because 1 + 2 = 3. This essentially tells us that if we were to going to try and factor $f(x)=3x^3-4x^2+5x+6$, we need to find just one linear factor of it first. To find which linear polynomial might work, we will utilize something called the rational root theorem. The rational root theorem is a quick way to see possible values of the constant to use in long division that may divide the two polynomials evenly.

Rational Root Theorem Given the equation for a polynomial in the form

$p(x)=a_n x^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+....+a_2x^2+a_1x+a_0$, where $a_0, a_1, a_2, ..., a_n$ are constant integer coefficients, we can find possible roots written as a fraction $x=\pm\frac{p}{q}$ in lowest terms given that:

- p is integer factors of the constant term a_0
- q is integer factors of the leading coefficient a_n

The root theorem is another theorem that helps with the division of polynomials. If we let *f(x)* be a polynomial of degree 1 or greater and let *a* be a number. The Division Algorithm tells us that $f(x)=q(x)\cdot(x-a)+r(x)$, where degree of *r(x)* is 0; hence *r(x)* = c, a constant. If *a* is a root of *f(x),* then $0=f(a)=q(a)\cdot(a-a)+c$. It follows that c = 0 and thus that $f(x)=q(x)\cdot(x-a)$ and *f(x)* has $(x-a)$ as a factor. Conversely, if c = 0, then a must be a root of *f(x).* This proves:

Root Theorem If *f(x)* is a polynomial with coefficients in a field, and if *a* is in the field with $f(a)=0$, then $x-a$ divides *f(x).*

Given the basics of the rational root theorem, we will now look at an example of how the theorem works using the example of x^2+5x-6. If we identify values for *p* and *q* we come up with

$$
\begin{array}{r}
x-1 \\
x+6\overline{)\,x^2+5x-6} \\
\underline{x^2+6x} \\
-1x-6 \\
\underline{-1x-6} \\
0
\end{array}
$$

$p = 1,6$ and $q = 1$. Therefore our possible values for x are $x = \pm \frac{1,6}{1}$. We end up with the following possible solutions: $x = 1, -1, 6, -6$. We can now pick one of these as our constant, for example, $x + 6$, and use long division to see if we have found something that divides $x^2 + 5x - 6$ with remainder zero. From the long division performed to the right, one can tell that $x + 6$ does divide $x^2 + 5x - 6$ evenly and the factorization is $(x+6)(x-1)$. The root theorem works because it essentially gives you the possible factors that could multiply out to give the constant of f(x) at the end. For instance, we knew that we could solve $x^2 + 5x - 6$ by factoring and doing so we would ask ourselves the factors of 6 that add up to 5. The factors that work are 1 and 6, and these were our possible values of x in the root theorem we used. The Root Theorem can be generalized to more complex theorems. The Horner Scheme is one that is helpful in eliminating roots at different stages, but that is beyond the scope of this paper.

Solving Polynomials

Now that we've discussed how to simplify polynomials, we need to extend this to the solution of equations involving polynomials. Most of us are familiar with some of the ways we can go about solving polynomial equations. When setting the polynomial equation equal to zero and solving, we are doing something called finding the roots of the polynomial. Graphically, this is where the graph of the polynomial crosses the x-axis. If we are solving for a linear function such as $f(x) = 2x + 4$, we can solve this a few different ways:

Solve using subtraction and division

$$f(x) = 2x + 4$$
$$0 = 2x + 4$$
$$-4 = 2x$$
$$-2 = x$$

Solve by factoring

$$f(x) = 2x + 4$$
$$0 = 2x + 4$$
$$0 = 2(x + 2)$$

Here we find that when x = -2, we have the solution!

If we are solving for a quadratic function such as $f(x) = x^2 + 6x + 5$, we again can solve this in a few different ways. We could solve by factoring, by graphing, by completing the square, or by using the quadratic formula. In this case, I will use completing the square to solve this function.

Solve by completing the square

$f(x) = x^2 + 6x + 5$

$x^2 + 6x + 5 = 0$ Set function equal to zero

$x^2 + 6x + \_\_ = -5$ Create a gap to complete the square

$x^2 + 6x + 9 = 4$ Take half of middle term, square, add to both sides

$x^2 + 6x + 9 = 4$

$(x+3)(x+3) = 4$ Factor left hand side

$(x+3)^2 = 4$ Rewrite left hand side

$\sqrt{(x+3)^2} = \sqrt{4}$ Take the square root of both sides

$(x+3) = \pm 2$

$x + 3 = 2 \qquad x + 3 = -2$ Set up two equations and solve

$x = -1 \qquad x = -5$

We obtain the solutions of x = -1 and x = -5, meaning that $x^2 + 6x + 5$ factors into $(x+5)(x+1)$.

It was previously mentioned that another way to solve quadratic equations was to use the quadratic formula. So why does the quadratic formula work and from where does it come? Its proof is given below.

$ax^2 + bx + c = 0$ $ax^2 + bx = -c$	We start with a general form of the equation and solve by completing the square in the steps below.
$x^2 + \frac{b}{a}x = \frac{-c}{a}$ $x^2 + \frac{b}{a}x = \frac{-c}{a}$	Begin by subtracting c from both sides and dividing by a.
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$	Complete the square by taking half of the middle term, squaring it, and adding it to both sides.
$\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$	Factor the left hand side
$\left(x + \frac{b}{2a}\right)^2 = \frac{-c}{a} + \frac{b^2}{(2a)^2}$	Square the far right term on the right side and rewrite the left hand side
$\left(x + \frac{b}{2a}\right)^2 = \frac{-c}{a} + \frac{b^2}{4a^2}$	Square the denominator on the far right hand side
$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$	Multiply the first term on the right side by $\frac{4a}{4a}$
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Rearrange terms since the right side has common denominators
$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$	Take the square root of both sides
$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a}$	Simplify the bottom term on the right hand side
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Subtract $\frac{b}{2a}$ from both sides and end with the quadratic formula!

The quadratic formula proves very helpful in solving a quadratic equation. It tells you exactly if a quadratic equation has 0, 1, or 2 real solutions, and this is indicated by observing the sign of the radicand, which is the number under the square root; in this context, the radicand is called the discriminant. If the discriminant is zero, there will be only one real solution. If the discriminant is negative, there are no real solutions. Finally, if the discriminant is positive, then there will be two real solutions to the quadratic equation. To understand why the quadratic equations can have

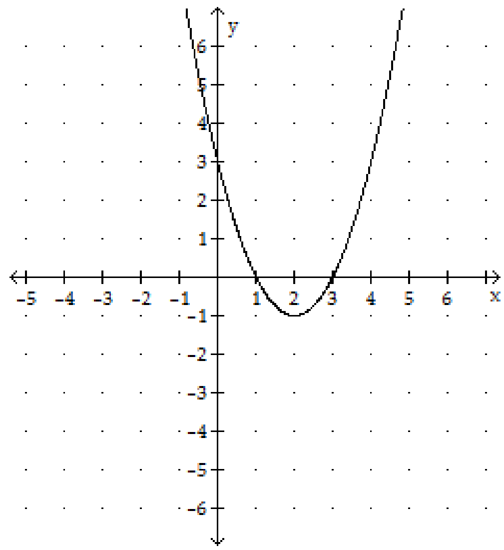
0, 1, or 2 real solutions, we will take a look at a few examples, namely $f(x) = x^2 - 4x + 3$,

$g(x) = x^2 - 2x + 1$, and $g(x) = x^2 - 2x + 10$.

$$f(x) = x^2 - 4x + 3$$

Solve by factoring

$f(x) = x^2 - 4x + 3$
$0 = x^2 - 4x + 3$
$0 = (x-3)\cdot(x-1)$

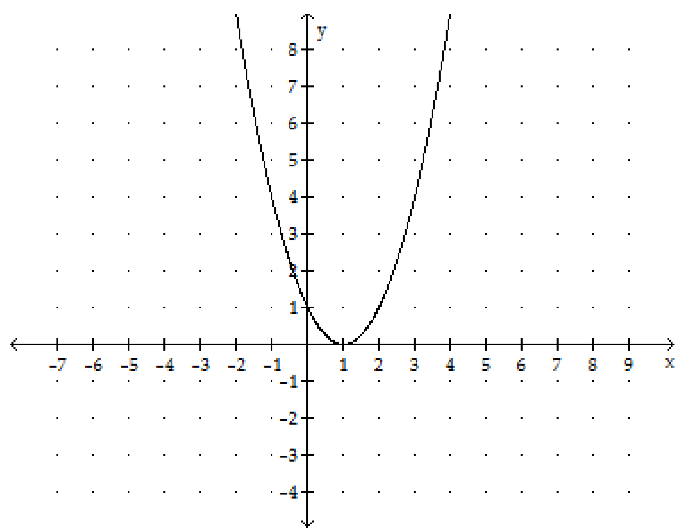


Here we find that when x = 3 and x = 1, we have our solutions! The graph to the right verifies the two solutions.

$$g(x) = x^2 - 2x + 1$$

Solve by factoring

$f(x) = x^2 - 2x + 1$
$0 = x^2 - 2x + 1$
$0 = (x-1)\cdot(x-1)$



Here we find that the solution is x = 1

From the graph to the right we can see there is only one root and it occurs at x = 1.

$$f(x) = x^2 - 4x + 3$$

Solve using quadratic formula

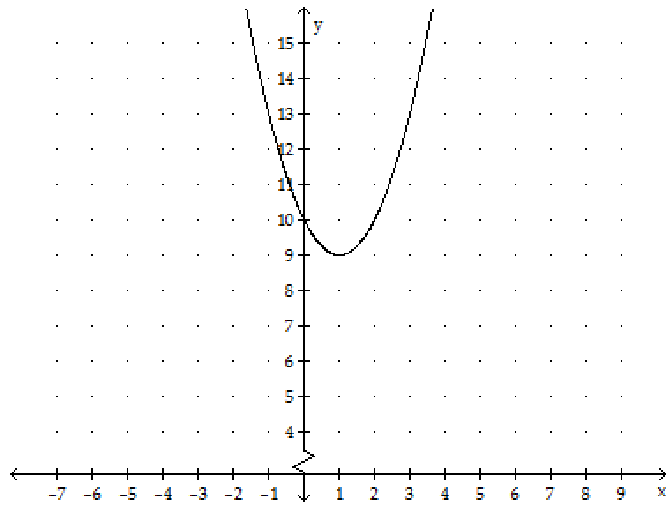
$$f(x) = x^2 - 2x + 10$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)}$$

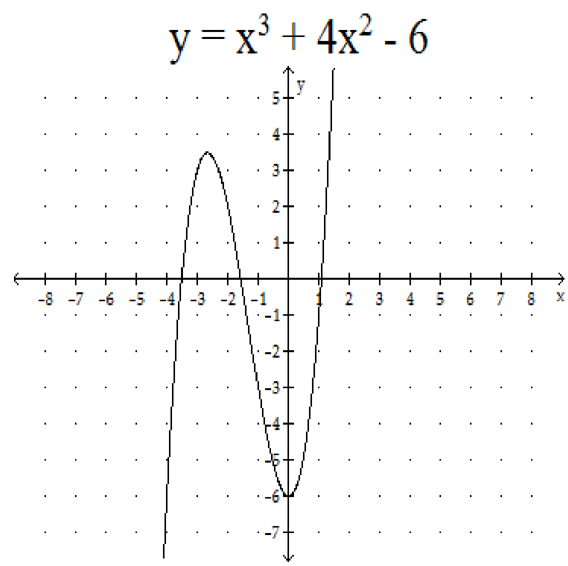
$$x = \frac{2 \pm \sqrt{4 - 40}}{2}$$

$$x = \frac{2 \pm \sqrt{-36}}{2}$$



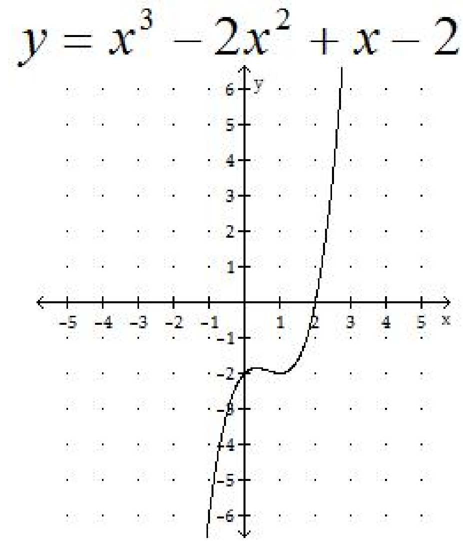
Here we find that we can't get a solution over the real numbers because we have a square root of a negative number. From the graph to the right we can see that we would expect no answer because the graph of the polynomial doesn't cross the x-axis.

So how do we solve a polynomial with a degree larger than 2? One of the ways is to use the rational root theorem as previously mentioned to find any rational root and thus obtain the factors of the polynomial. Alternatively, we could use a graphing calculator to estimate the roots. For example if you are given the following equation and graph to the right, the rational root theorem gives us only $x = \pm 1,\ \pm 2,\ \pm 3,\ \pm 6$ and none of these work. So we can approximate the roots by using the table function on a graphing calculator. You begin by placing the equation in the y = window and then looking at the



x	y	
-3.52	-.0526	← zero occurs between here
-3.51	.03685	
-1.58	.04129	← zero occurs between here
-1.57	-.0103	
1.08	-.0747	← zero occurs between here
1.09	.04743	

graph you can get an estimate of the x value where the line crosses the x-axis. Once you have an idea of the x value, you can use the table feature to pinpoint the exact value more closely. The table allows you to change how closely you would like to analyze the function. The top table to the right is an example of how you could “zoom” in on the x values of the equation graphed above. We could continue to make the x values really small until we got y to be essentially zero and this would give us the three roots of this equation. The table to the right shows how we can “zoom” in and get really close the values of x that make y zero. What this now tells us is that our solutions are approximately x = -3.515, -1.575, and 1.085. This means that if we factored $x^3 + 4x^2 - 6$ we would get approximately $(x+3.515)(x+1.575)(x-1.085)$, which is approximately the complete factorization of the polynomial.

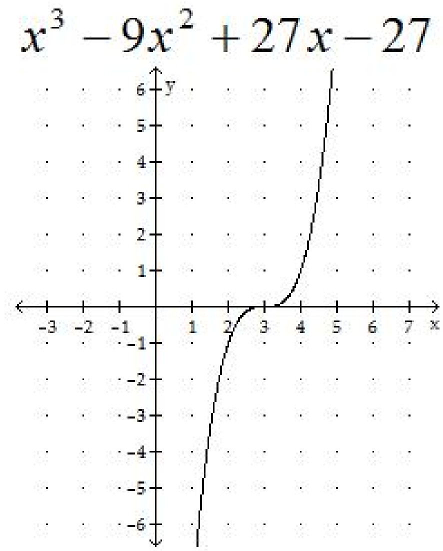


The previous function was a case in which a polynomial of degree three gave us exactly three real roots. If we look at another polynomial of degree three, we can see from the graph we have only one real root and it occurs at x = 2. The rational root theorem would also give us x = 2 as a possible root of the polynomial. If we factor out $(x-2)$, we get:

$$x^3 - 2x^2 + x - 2 = (x-2)(x^2+1)$$

From this it is not intuitively obvious, but (x^2+1) can factor into the complex factors of $(x+\sqrt{-1})\cdot(x-\sqrt{-1})$, more commonly written as $(x+i)\cdot(x-i)$. Since this function has two complex roots, there is only one real root. This example brings us to an interesting point that should be made about complex roots. In order to achieve a negative square root, the factors of a polynomial with real coefficients must appear in conjugate pairs. Therefore, complex roots of polynomials with real coefficieints will always appear in conjugate pairs. This means that if one complex factor of a polynomial is $\left(x+\sqrt{-5}\right)$, or $(x+5i)$, then there must be another factor of the polynomial that is $\left(x-\sqrt{-5}\right)$, or equivalently, $(x-5i)$.

In similar fashion, the degree doesn't necessarily tell exactly how many different roots exist. Take for instance the equation and graph at the right. If we were to factor this, we could divide by $(x-3)$and get:



$$x^3-9x^2+27x-27=(x-3)(x^2-6x+9)$$

$$x^3-9x^2+27x-27=(x-3)(x-3)(x-3)$$

This shows that all three roots are real solutions and they are exactly the same. This concept of having roots repeat is called algebraic multiplicity. It can occur with complex roots as well, but the degree of the polynomial would have to be at least four since the complex roots will come in conjugate pairs. Thus you would have to have two pairs of each complex factor.

So how do we know which functions have roots and how many they have? We now have reached a point where we can now introduce the Fundamental Theorem of Algebra. The development of the FTA in early years only allowed real values and so the FTA was not very relevant. It was Girolamo Cardano (1501 – 1576) in 1545 that realized he could get answers that were not real numbers and he began the idea of complex solutions. In 1572, Rafael Bombelli (1526 – 1572) set about to create a rule for defining these complex numbers. Albert Girard, a Flemish mathematician, was the first to claim that there are always *n* solutions to a polynomial of degree *n* in 1629 in his book *L'invention en algebra*. In 1637, Descartes conjectured that for every polynomial of degree n, there were n roots. He was unable to extend his discovery much past this because he noted the imagined roots did not have a real quantity. There were many other mathematicians who tackled the FTA, but it was Gauss who is credited with the first proof in his doctoral thesis of 1799. He spotted errors in previous proofs because many earlier attempts assumed too many things. Gauss later published other proofs and made his fourth and final proof in 1849, fifty years after his first proof. This time he made his proof complete and accurate. The FTA has a few different versions, but it is most commonly presented as:

A polynomial equation of degree *n* with complex coefficients can be written as a product of n linear (first degree) factors.

The fundamental theorem does not tell how to factor a polynomial nor does it tell how to solve for the roots, which can be easily solved for from the 1st degree factors. To factor a polynomial we need to use some of the previously mentioned methods, such as analyzing the graphs and trying to pull out a linear factor or using the root theorem. For example, if we want to factor $x^3 + x^2$, then it would factor into $x^2(x+1) = x \cdot x(x+1)$ and since we have three linear factors, we cannot factor anymore. Another example shows how $x^3 + x^2 + x$ can factor into $x(x^2 + x + 1)$. Here x is linear and $x^2 + x + 1$ is irreducible over the real field. This is why the fundamental theorem includes the "complex roots" part. Thus over the complex field, $x^2 + x + 1$ can actually be reduced to two linear complex factors. Through the use of the quadratic formula we can find these to be: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \dfrac{-1 \pm \sqrt{-3}}{2}$ which means x =

$$\frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$$

and we have factors of $\left(x - \dfrac{-1+\sqrt{-3}}{2}\right) \cdot \left(x - \dfrac{-1-\sqrt{-3}}{2}\right)$ which simplifies to

$$\left(x - \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)\right) \cdot \left(x - \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)\right).$$

From this these examples, the FTA makes more sense and should help one understand why a polynomial equation of degree *n* with complex coefficients has *n* complex roots.

No mathematician has been able to come with an exact formula for how to factor polynomials. Much of the factoring done in high school algebra is to know the patterns that exist and then use the guess and check method. We have techniques to help factor linear functions (e.g. 2x + 4) and quadratic function (e.g. $x^2 + 4x + 3$). There are even formulas found to find the roots of quadratic, cubic, and quartic functions, which in turn would allow us to factor these. The cubic and quartic formulas are much more complicated than the quadratic formula and would take too much time to explain in the content of this paper. However, there does not exist an algorithm for factoring a polynomial of degree five or higher. In fact, the mathematician Galois proved that there will never be a formula to solve the general polynomial of degree five or higher. His work is known as Galois Theory.

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